Jet cross sections at NNLO via local subtraction

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DESY

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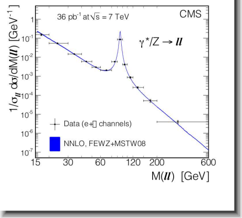
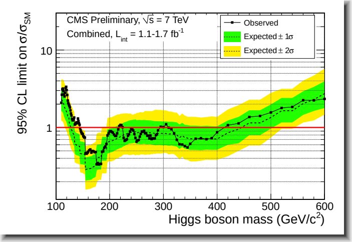
Differential NNLO

A young and promising field in the LHC era

- less than a decade old
- starting from simple decays and single production processes
- moved to/moving towards complicated jet production and pair production processes at colliders
- already a significant impact on phenomenology at collider experiments





$$\boxed{\alpha_{\mathrm{s}} = 0.1175 \pm 0.0020\ (\text{exp}) \pm 0.0015\ (\text{theo})}$$

Outline

Introduction

Basics of subtraction

Local subtraction at NNLO

The tedious part: integrating the counterterms

Integrated approximate cross sections

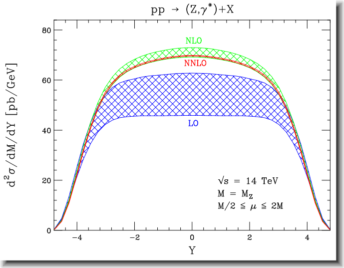
Outlook

Introduction

Motivation - why NNLO?

Precision QCD requires computations beyond NLO in certain cases

- NLO corrections are large:
  Higgs production from gluon fusion in hadron collisions
- the main source of uncertainty in extracting info from data is due to NLO theory:
  α_s measurements
- reliable error estimate is needed:
  precise measurement of parton luminosities



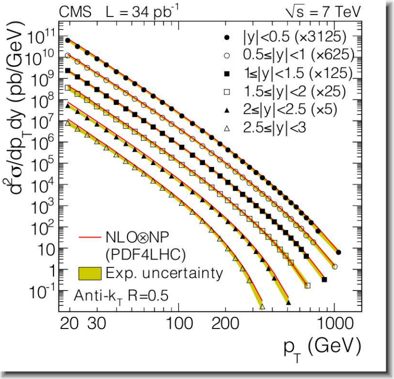
(Anastasiou, Dixon, Melnikov, Petriello, Phys. Rev. **D69** (2004) 094008.)

In short, NNLO is relevant when NLO fails to do its job

Motivation - why jets at NNLO?

Jets are essential analysis tools at LHC: precise understanding is needed

- status at LHC: looks good. . .



(CMS Collaboration, Phys. Rev. Lett. **107** (2011) 132001.)

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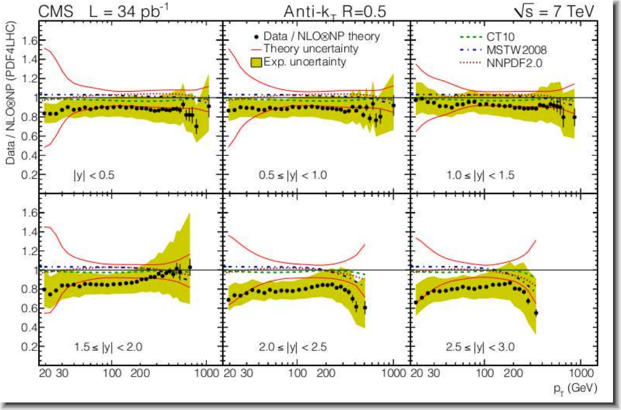

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Motivation - why jets at NNLO?

Jets are essential analysis tools at LHC: precise understanding is needed

- status at LHC: looks good... but have a closer look!
- jet energy scale uncertainty of $\simeq$ 5-10 % warrants precision physics
- precision prediction for 'standard candles': inclusive jet, V + jet, ...
- missing piece for precise determination of pdf's
- NLO is effectively LO: energy distribution inside jet cones, jet $p_\perp$ asymmetry, ...

Status

Processes measured to few precent accuracy

- $e^+e^- \to 3j$
- $ep \to (2+1)j$
- $pp \to j + X$
- $pp \to V$
- $pp \to V + j$
- $pp \to t\bar{t}$

Processes with potentially large radiative corrections

- $pp \to H$
- $pp \to H + j$
- $pp \to VV$

Status

Processes measured to few precent accuracy

- $e^+e^- \to 3j$ ✔
- $ep \to (2+1)j$ ✘
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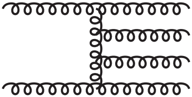
Processes with potentially large radiative corrections

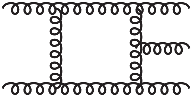
- $pp \to H$ ✔
- $pp \to H + j$ ✘
- $pp \to VV$ ✔ $(VV = \gamma\gamma)$

NNLO ingredients

A generic m-jet cross section at NNLO involves



- Tree-level squared matrix elements
  - with $m+2$ parton kinematics
  - known from LO calculations
  - ‘doubly-real’ contribution (RR)



- One-loop squared matrix elements
  - with $m+1$ parton kinematics
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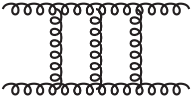

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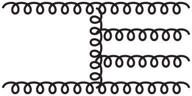

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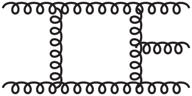

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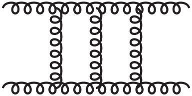


Assuming we know the relevant matrix elements, can we use those matrix elements to compute cross sections?

The problem - IR singularities

Consider the NNLO correction to a generic *m*-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m} \,.$$

Doubly-real

- $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2}$
- Tree MEs with $m+2$-parton kinematics
- kin. singularities as one or two partons unresolved: up to $\mathrm{O}(\epsilon^{-4})$ poles from PS integration
- no explicit ϵ poles

Real-virtual

- $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1}$
- One-loop MEs with $m+1$-parton kinematics
- kin. singularities as one parton unresolved: up to $\mathrm{O}(\epsilon^{-2})$ poles from PS integration
- explicit ϵ poles up to $\mathrm{O}(\epsilon^{-2})$

Doubly-virtual

- $\mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$
- One- and two-loop MEs with *m*-parton kinematics
- kin. singularities screened by jet function: PS integration finite
- explicit ϵ poles up to $\mathrm{O}(\epsilon^{-4})$

The problem - IR singularities

Consider the NNLO correction to a generic *m*-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m} \,.$$

THE KLN THEOREM

Infrared singularities cancel between real and virtual quantum corrections at the same order in perturbation theory, for sufficiently inclusive (i.e. IR safe) observables.

HOWEVER

How to make this cancellation explicit, so that the various contributions can be computed numerically? Need a method to deal with implicit poles.

Approaches

Sector decomposition

(Binoth, Heinrich; Anastasiou, Melnikov, Petriello; Czakon)

- extract ϵ poles of each contribution (RR, RV, VV) separately by expanding the integrand in distributions
- resulting expansion coefficients are finite multi-dimensional integrals, integrate numerically
- cancellation of poles numerical, depends on observable
- first method to yield physical results, but can it handle complicated final states?

Subtraction

(Catani, Grazzini; Cieri, Ferrera, de Florian; Gehrmann, Gehrmann-De Ridder, Glover; Weinzierl; Del Duca, Trócsányi, GS)

- rearrange the poles between real and virtual contributions by subtracting and adding back suitable approximate cross sections
- cancellation of explicit ϵ poles achieved analytically, remaining PS integrals are finite
- nice properties (generality, efficiency) expected form experience at NLO
- definition of subtraction terms is not unique, hence several approaches: $q_\perp$, antenna, local

Approaches

Sector decomposition

(Binoth, Heinrich, Anastasiou, Dixon, Melnikov, Petriello, Czakon)

- ✔ first method to yield physical cross sections
- ✔ cancellation of divergences fully numerical
- ✘ cancellation of poles also, and depends on jet function
- ✘ can it handle complicated final states?

$q_\perp$ subtraction

(Catani, Grazzini, Cieri, Ferrera, de Florian, Tramontano)

- ✔ exploits universal behavior of $q_\perp$ distribution at small $q_\perp$
- ✔ efficient and fully exclusive calculation
- ✘ limited scope: applicable only to production of massive colorless final states in hadron collisions

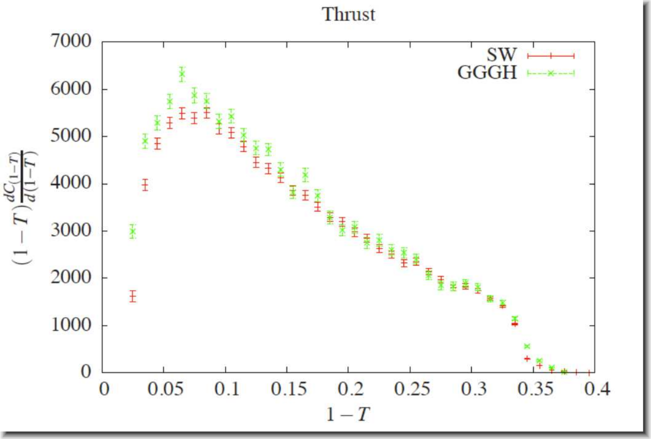
Antenna subtraction

(Gehrmann, Gehrmann-De Ridder, Glover, Heinrich, Weinzierl)

- ✔ successfully applied to $e^+e^- \to 2, 3j$
- ✔ analytic integration of antennae over unresolved phase space is understood
- ✘ counterterms are nonlocal
- ✘ treatment of color is implicit
- ✘ cannot cut factorized phase space

Approaches

Is the agreement between antenna implementations satisfactory? (Weinzierl)



Why a new scheme?

Goal: devise a subtraction scheme with

- general and explicit expressions, including color
  (view towards automation, color space notation is used)
- fully local counterterms, taking account of all color and spin correlations
  (mathematical rigor, efficiency)
- option to constrain subtractions to near singular regions
  (efficiency, important check)
- very algorithmic construction
  (valid at any order in perturbation theory)

Basics of subtraction

Subtraction - a caricature

Want to evaluate (at $\epsilon \to 0$)

$$\sigma = \int_0^1 \mathrm{d}\sigma^{\mathrm{R}}(x) + \sigma^{\mathrm{V}} \qquad \text{where} \qquad \begin{aligned} \mathrm{d}\sigma^{\mathrm{R}}(x) &= x^{-1-\epsilon} R(x) \\ R(0) &= R_0 < \infty \\ \sigma^{\mathrm{V}} &= R_0/\epsilon + V \end{aligned}$$

- define the counterterm

$$\mathrm{d}\sigma^{\mathrm{R,A}}(x) = x^{-1-\epsilon} R_0$$

- use it to reshuffle singularities between R and V contributions

$$\begin{aligned} \sigma &= \int_0^1 \left[\mathrm{d}\sigma^{\mathrm{R}}(x) - \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} + \left[\sigma^{\mathrm{V}} + \int_0^1 \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} \\ &= \int_0^1 \left[\frac{R(x) - R_0}{x^{1+\epsilon}}\right]_{\epsilon=0} + \left[\frac{R_0}{\epsilon} + V - \frac{R_0}{\epsilon}\right]_{\epsilon=0} \\ &= \int_0^1 \frac{R(x) - R_0}{x} + V \end{aligned}$$

The last integral is finite, computable with standard numerical methods.

The issue of locality

In a rigorous mathematical sense, the cancellation of both kinematical singularities and ϵ-poles must be local. I.e. the subtraction term must have the following general properties

- it must match the singularity structure of (singly- and doubly-) real emissions pointwise, in *d* dimensions
- its integrated form must be combined with the (real- and doubly-) virtual cross section explicitly, before phase space integration; ϵ-poles must cancel point by point

What about singular terms in the real emission cross section that cancel upon phase space integration (e.g. azimuthal correlations in gluon splitting)?

- they cancel upon integration in *d* dimensions, the corresponding four dimensional integrals are ill-defined
- it is mandatory to treat these terms, since naive numerical integration (in four dimensions) can give any result whatsoever
- however, can be treated with methods other than strict local subtraction, e.g. auxiliary phase space slicing (as in antenna subtraction)

A more efficient subtraction scheme?

Want to evaluate (at $\epsilon \to 0$)

$$\sigma = \int_0^1 \mathrm{d}\sigma^{\mathrm{R}}(x) + \sigma^{\mathrm{V}} \qquad \text{where} \qquad \begin{aligned} \mathrm{d}\sigma^{\mathrm{R}}(x) &= x^{-1-\epsilon} R(x) \\ R(0) &= R_0 < \infty \\ \sigma^{\mathrm{V}} &= R_0/\epsilon + V \end{aligned}$$

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- define the counterterm to be nonzero only near singular region

$$\mathrm{d}\sigma^{\mathrm{R,A}}(x) = x^{-1-\epsilon} R_0\, \Theta(x_0 - x)$$

- use it to reshuffle singularities between R and V contributions

$$\begin{aligned} \sigma &= \int_0^1 \left[\mathrm{d}\sigma^{\mathrm{R}}(x) - \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} + \left[\sigma^{\mathrm{V}} + \int_0^1 \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} \\ &= \int_0^1 \left[\frac{R(x) - R_0\Theta(x_0 - x)}{x^{1+\epsilon}}\right]_{\epsilon=0} + \left[\frac{R_0}{\epsilon} + V - \frac{R_0}{\epsilon} + R_0 \log x_0 + \mathrm{O}(\epsilon^1)\right]_{\epsilon=0} \\ &= \int_0^1 \frac{R(x) - R_0\Theta(x_0 - x)}{x} + V + R_0 \log x_0 \end{aligned}$$

The last integral is finite, computable with standard numerical methods.

A more efficient subtraction scheme?

It is sufficient to perform subtraction only near the singular region

- ✓ gain in efficiency: subtraction term only needs to be computed over a fraction of phase space
- ✓ strong check: final result is independent of value of phase space cut
- ✗ analytical integration of subtraction term more difficult (extra scale involved)

Local subtraction at NNLO

Structure of the NNLO correction

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

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$$\sigma^{\rm NNLO}_{m+1} = \int_{m+1}\left\{\left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}\right]J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}\right)^{\rm A_1}\right]J_m\right\}$$

$$\sigma^{\rm NNLO}_{m} = \int_{m}\left\{{\rm d}\sigma^{\rm VV}_{m} + \int_2\left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2}\right] + \int_1\left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}\right)^{\rm A_1}\right]\right\}J_m$$

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1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
3. $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$ accounts for the overlap of $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$

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$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
3. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2}$ accounts for the overlap of $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2}$
4. $\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$

Structure of the NNLO correction

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
3. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2}$ accounts for the overlap of $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2}$
4. $\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$
5. $\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}\right)^{\mathrm{A}_1}$ regularizes the singly-unresolved limit of $\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$

Defining a subtraction scheme

Strategy: IR limits are process independent and known

1. Start from defining the subtraction terms based on IR limit formulae
   - they are trivially general, explicit and local
   - done some time ago (2006) for colorless initial states
2. Worry about integrating them later
   - since this is *in principle* a very narrowly defined problem, given 1.
   - but in practice is very cumbersome, due to lack of technology

Defining a subtraction scheme

The following three problems must be addressed

1. Matching of limits to avoid multiple subtraction in overlapping singular regions of PS. Easy at NLO: collinear limit + soft limit - collinear limit of soft limit.

$$\mathbf{A}_1 |\mathcal{M}_{m+1}^{(0)}|^2 = \sum_i \left[\sum_{i \neq r} \frac{1}{2} \mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i \neq r} \mathbf{C}_{ir} \mathbf{S}_r \right] |\mathcal{M}_{m+1}^{(0)}|^2$$

2. Extension of IR factorization formulae over full PS using momentum mappings that respect factorization and delicate structure of cancellations in all limits.

$$\{p\}_{m+1} \xrightarrow{r} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+1}(\{p\}_{m+1}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m; Q)[\mathrm{d}p_{1,m}]$$

$$\{p\}_{m+2} \xrightarrow{r,s} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+2}(\{p\}_{m+2}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m; Q)[\mathrm{d}p_{2,m}]$$

3. Integration of the counterterms over the phase space of the unresolved parton(s).

The need for extension

IR limit formulae are only well-defined in the strict limit. E.g.

- collinear: $\mathbf{C}_{ir}$ is a symbolic operator that takes the $p_i || p_r$ limit

$$\mathbf{C}_{ir} |\mathcal{M}^{(0)}_{m+2}(p_i, p_r, \ldots)|^2 = 8\pi\alpha_{\rm s}\mu^{2\epsilon} \frac{1}{s_{ir}} \hat{P}_{f_i f_r}(z_i, z_r, k_\perp; \epsilon) \otimes |\mathcal{M}^{(0)}_{m+1}(p_{ir}, \ldots)|^2$$

- soft: $\mathbf{S}_r$ is a symbolic operator that takes the $p_r \to 0$ limit

$$\mathbf{S}_r |\mathcal{M}^{(0)}_{m+2}(p_r, \ldots)|^2 = -8\pi\alpha_{\rm s}\mu^{2\epsilon} \sum_{i,k} \frac{1}{2} \mathcal{S}_{ik}(r) |\mathcal{M}^{(0)}_{m+1,(i,k)}(\xcancel{p_r}, \ldots)|^2$$

NOTICE

- momenta in factorized ME's on the r.h.s. conserve momentum and/or mass shell conditions only in the strict limit
- arguments of AP splitting functions, e.g. momentum fractions z_i, z_r and transverse momentum $k_\perp$ are only defined in the strict limit

HENCE

- must specify precisely momenta entering factorized ME's away from limit
- must define z_i, z_r and $k_\perp$ away from limit

Defining a subtraction scheme

Specific issues at NNLO

1. Matching is cumbersome if done in a brute force way. However, an efficient solution that works at any order in PT is known.
2. Extension is delicate. E.g. counterterms for singly-unresolved real emission (unintegrated and integrated) must have universal IR limits. This is not guaranteed by QCD factorization.
3. Choosing the counterterms such that integration is (relatively) straightforward generally conflicts with the delicate cancellation of IR singularities.

NNLO subtraction terms - an example

MESSAGE

- Subtraction terms are defined completely explicitly for any number of jets.

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$
$$\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i},z_{i,r},k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j},z_{j,s},k_{\perp,j,s};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})\rangle$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{\boxed{s_{ir}s_{js}}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$

$$\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i}, z_{i,r}, k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j}, z_{j,s}, k_{\perp,j,s};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})\rangle$$

➠ collinear poles: $\boxed{s_{ir}s_{js}}$

$$s_{kl} = 2p_k \cdot p_l\,, \qquad k,l = i,r \quad \text{or} \quad j,s$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$
$$\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})|\boxed{\hat{P}_{f_i f_r}(z_{r,i},z_{i,r},k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j},z_{j,s},k_{\perp,j,s};\epsilon)}|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})\rangle$$

- Altarelli-Parisi splitting functions: $\boxed{\hat{P}_{f_i f_r}\hat{P}_{f_j f_s}}$

$$z_{k,l} = \frac{y_{kQ}}{y_{(kl)Q}}\,, \qquad k_{\perp,k,l}^{\mu} = \zeta_{k,l}p_l^{\mu} - \zeta_{l,k}p_k^{\mu} + \zeta_{kl}\tilde{p}_{kl}^{\mu}\,, \qquad k,l = i,r \quad \text{or} \quad j,s$$

with

$$\zeta_{k,l} = z_{k,l} - \frac{y_{kl}}{\alpha_{kl}y_{(kl)Q}}\,, \qquad \zeta_{kl} = \frac{y_{kl}}{\alpha_{kl}y_{\widetilde{kl}Q}}(z_{l,k}-z_{k,l})$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$
$$\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i}, z_{i,r}, k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j}, z_{j,s}, k_{\perp,j,s};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})\rangle$$

- mapped momenta: $\{\tilde{p}\}^{(ir;js)} = \{\tilde{p}_1, \ldots, \tilde{p}_{ir}, \ldots, \tilde{p}_{js}, \ldots, \tilde{p}_{m+2}\}_m$

$$\tilde{p}_{kl}^{\mu} = \frac{1}{1-\alpha_{ir}-\alpha_{js}}(p_k^{\mu}+p_l^{\mu}-\alpha_{kl}Q^{\mu}), \qquad k,l = i,r \quad \text{or} \quad j,s$$

$$\tilde{p}_n^{\mu} = \frac{1}{1-\alpha_{ir}-\alpha_{js}}p_n^{\mu}, \qquad n \neq i,r,j,s$$

with

$$\alpha_{kl} = \frac{1}{2}\left[y_{(kl)Q} - \sqrt{y_{(kl)Q}^2 - 4y_{kl}}\right], \qquad k,l = i,r \quad \text{or} \quad j,s$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\boxed{\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})}$$

$$\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i},z_{i,r},k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j},z_{j,s},k_{\perp,j,s};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})\rangle$$

➠ constrain subtraction to near singular region: $\boxed{\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})}$

$$0 < \alpha_0 \leq 1\,, \qquad \alpha_0 = 1\text{: subtract over full phase space}$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}} \boxed{(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}} \Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$

$$\times \langle \mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)}) | \hat{P}_{f_i f_r}(z_{r,i}, z_{i,r}, k_{\perp,i,r}; \epsilon) \hat{P}_{f_j f_s}(z_{s,j}, z_{j,s}, k_{\perp,j,s}; \epsilon) | \mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)}) \rangle$$

➠ make integrated counterterm m-independent: $\boxed{(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}}$

$$d(m;\epsilon) = 2m(1-\epsilon) - 2d_0\,, \qquad d_0 = D_0 + d_1\epsilon\,, \qquad D_0 \geq 2$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2}$ we find

$$\mathcal{C}^{(0,0)}_{ir;js}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$
$$\times \langle \mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i},z_{i,r},k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j},z_{j,s},k_{\perp,j,s};\epsilon)|\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir;js)})\rangle$$

The complete approximate cross section is a sum of such terms

$$\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} = \mathrm{d}\phi_m[\mathrm{d}p_2]\boldsymbol{\mathcal{A}}_2|\mathcal{M}^{(0)}_{m+2}|^2$$

where

$$\boldsymbol{\mathcal{A}}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \sum_{s\neq r}\Bigg\{\sum_{i\neq r,s}\Bigg[\frac{1}{6}\mathcal{C}^{(0,0)}_{irs} + \sum_{j\neq i,r,s}\frac{1}{8}\mathcal{C}^{(0,0)}_{ir;js} + \frac{1}{2}\mathcal{CS}^{(0,0)}_{ir;s}\Bigg] + \frac{1}{2}\mathcal{S}^{(0,0)}_{rs}$$
$$- \sum_{i\neq r,s}\Bigg[\frac{1}{2}\mathcal{C}_{irs}\mathcal{CS}^{(0,0)}_{ir;s} + \sum_{j\neq i,r,s}\frac{1}{2}\mathcal{C}_{ir;js}\mathcal{CS}^{(0,0)}_{ir;s} + \frac{1}{2}\mathcal{C}_{irs}\mathcal{S}^{(0,0)}_{rs} + \mathcal{CS}_{ir;s}\mathcal{S}^{(0,0)}_{rs}$$
$$- \sum_{j\neq i,r,s}\frac{1}{2}\mathcal{C}_{ir;js}\mathcal{S}^{(0,0)}_{rs} - \mathcal{C}_{irs}\mathcal{CS}_{ir;s}\mathcal{S}^{(0,0)}_{rs}\Bigg]\Bigg\}$$

NNLO subtraction terms - general features

Based on universal IR limit formulae

- Altarelli-Parisi splitting functions, soft currents (tree and one-loop, triple AP functions)
- simple and general procedure for matching of limits using physical gauge
- extension based on momentum mappings that can be generalized to any number of unresolved partons

Fully local in color $\otimes$ spin space

- no need to consider the color decomposition of real emission ME's
- azimuthal correlations correctly taken into account in gluon splitting
- can check explicitly that the ratio of the sum of counterterms to the real emission cross section tends to unity in any IR limit

Straightforward to constrain subtractions to near singular regions

- gain in efficiency
- independence of physical results on phase space cut is strong check

Given completely explicitly for any process with non colored initial state

The tedious part: integrating the counterterms

Basic setup

Momentum mappings used to define the counterterms

$$\{p\}_{n+p} \xrightarrow{R} \{\tilde{p}\}_n$$

- implement exact momentum conservation
- recoil distributed democratically (can be generalized to any p)
- different collinear and soft mappings (R labels precise limit)
- exact factorization of phase space

$$\mathrm{d}\phi_{n+p}(\{p\};Q) = \mathrm{d}\phi_n(\{\tilde{p}\}_n^{(R)};Q)[\mathrm{d}p_{p,n}^{(R)}]$$

Counterterms are products (in color and spin space) of

- factorized ME's independent of variables in $[\mathrm{d}p_{p,n}^{(R)}]$
- singular factors (AP functions, soft currents), to be integrated over $[\mathrm{d}p_{p,n}^{(R)}]$

Strategy for computing the integrals

- explicit parametrization of factorized phase space leads to parametric integral representations
- evaluate the parametric integrals

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\rm NNLO} = \sigma^{\rm RR}_{m+2} + \sigma^{\rm RV}_{m+1} + \sigma^{\rm VV}_{m} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m}$$

each integrable in four dimensions

$$\sigma^{\rm NNLO}_{m+2} = \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\rm NNLO}_{m+1} = \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$\sigma^{\rm NNLO}_{m} = \int_{m} \left\{ {\rm d}\sigma^{\rm VV}_{m} + \int_2 \left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \boxed{\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} \right] + \int_1 \left[\boxed{\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1}} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

➠ tree-level and one-loop singly-unresolved integrals

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\rm NNLO} = \sigma^{\rm RR}_{m+2} + \sigma^{\rm RV}_{m+1} + \sigma^{\rm VV}_{m} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m}$$

each integrable in four dimensions

$$\sigma^{\rm NNLO}_{m+2} = \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\rm NNLO}_{m+1} = \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$\sigma^{\rm NNLO}_{m} = \int_{m} \left\{ {\rm d}\sigma^{\rm VV}_{m} + \int_2 \left[\boxed{{\rm d}\sigma^{\rm RR,A_2}_{m+2}} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

Phase space integrals - an example

MESSAGE

- The integral is (very) difficult, but the result is numerically (very) simple.

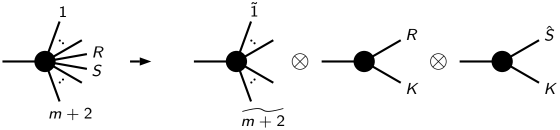
Phase space integrals - an example

Abelian soft-double soft counterterm: among many others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_{12}}$ we find

$$\left(\mathcal{S}_t\mathcal{S}_{rt}^{(0)}\right)^{\mathrm{ab}} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \sum_{i,j,k,l} \frac{1}{8}\mathcal{S}_{\hat{i}\hat{k}}(\hat{r})\mathcal{S}_{jl}(t)|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$
$$\times\,(1-y_{tQ})^{d_0'-m(1-\epsilon)}(1-y_{\hat{r}Q})^{d_0'-m(1-\epsilon)}\Theta(y_0-y_{tQ})\Theta(y_0-y_{\hat{r}Q})$$

The set of m momenta, $\{\tilde{p}\}$, is obtained by an iterated mapping which leads to an exact factorization of phase space

$$\{p\}_{m+2} \xrightarrow{\mathsf{S}_t} \{\hat{p}\}_{m+1} \xrightarrow{\mathsf{S}_{\hat{r}}} \{\tilde{p}\}\;:\quad \mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_m(\{\tilde{p}\};Q)[\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]$$



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Then we must compute

$$\int [\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]\mathcal{S}_t\mathcal{S}_{rt}^{(0)} \equiv \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i,k,j,l}[\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl}|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$

where $[\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl} \equiv [\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl}(p_i, p_k, p_j, p_l, \epsilon, y_0, d_0')$ is a kinematics dependent function.

Abelian soft-double soft integral

For simplicity, consider the terms in the sum where $j = i$ and $l = k$: $[\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikik}$. Kinematical dependence is through $\cos\chi_{ik} = \measuredangle(p_i, p_k)$, we set $\cos\chi_{ik} = 1 - 2Y_{ik,Q}$.

Using angles and energies in some specific Lorentz frame to parametrize the factorized phase space measures, $[\mathrm{d}\widehat{p}_{1,m}]$ and $[\mathrm{d}p_{1,m+1}]$, we find that $[\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikik}$ is proportional to

$$\begin{aligned}\mathcal{I}_{\mathcal{S}}^{(11)}(Y_{ik,Q}; \epsilon, y_0, d_0') &= -\frac{4\Gamma^4(1-\epsilon)}{\pi\Gamma^2(1-\epsilon)}\,\frac{B_{y_0}(-2\epsilon, d_0'+1)}{\epsilon}\,Y_{ik,Q}\int_0^{y_0}\mathrm{d}y\,y^{-1-2\epsilon}(1-y)^{d_0'-1+\epsilon}\\ &\times\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1}\mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon}\big[f(\vartheta,\varphi;0)\big]^{-1}\big[f(\vartheta,\varphi;Y_{ik,Q})\big]^{-1}\\ &\times\big[Y(y,\vartheta,\varphi;Y_{ik,Q})\big]^{-\epsilon}{}_2F_1\big(-\epsilon,-\epsilon,1-\epsilon,1-Y(y,\vartheta,\varphi;Y_{ik,Q})\big)\end{aligned}$$

where

$$f(\vartheta,\varphi;Y_{ik,Q}) = 1 - 2\sqrt{Y_{ik,Q}(1-Y_{ik,Q})}\sin\vartheta\cos\varphi - (1-2Y_{ik,Q})\chi\cos\vartheta$$

$$Y(y,\vartheta,\varphi;\chi) = \frac{4(1-y)Y_{ik,Q}}{[2(1-y)+y\,f(\vartheta,\varphi;0)][2(1-y)+y\,f(\vartheta,\varphi;Y_{ik,Q})]}$$

Abelian soft-double soft integral

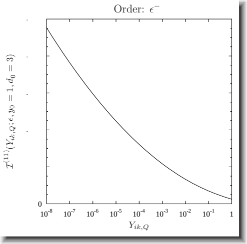
This integral is equal to

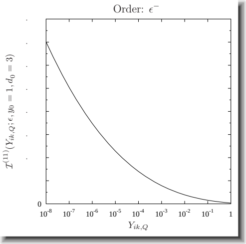
$$\mathcal{I}_{\mathcal{S}}^{(11)}(Y_{ik,Q};\epsilon,y_0,d_0') = \frac{1}{\epsilon^4} - 2\left[\ln(Y_{ik,Q}) + \Sigma(y_0,D_0') + \Sigma(y_0,D_0'-1)\right]\frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2})$$

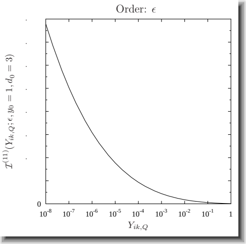
where $D_0' = d_0'|_{\epsilon=0}$ and the dependence on the PS cut parameter, y_0, enters in

$$\Sigma(z,N) = \ln z - \sum_{k=1}^{N} \frac{1-(1-z)^k}{k}$$

Higher order expansion coefficients computed numerically $(y_0 = 1,\ D_0' = 3)$







Analytical vs. numerical

As a matter of principle

- Rigorous proof of cancellation of IR poles requires poles of integrated counterterms in analytical form.
- Analytical forms are fast and accurate compared to numerical ones.

However

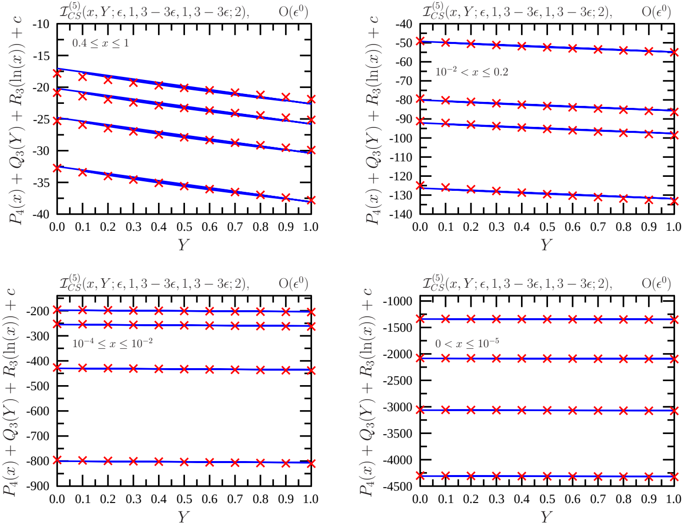
- Analytical results show (in all cases where they are available) that integrated counterterms are smooth functions of kinematic variables.

Hence

- Numerical forms of integrated counterterms are sufficient for practical purposes. Final results can be conveniently given by interpolating tables or approximating functions computed once and for all. Thus, efficient implementation is possible even if the full analytical calculation is not feasible or practical (e.g. finite parts of integrated counterterms).
- In particular, suitable approximating functions may be obtained by fitting.

Example of approximation by fitting

Doubly-unresolved soft-collinear master integral $\mathcal{I}^{(5)}_{\mathcal{CS}}(x, Y; \epsilon)$



Phase space integrals - methods

Several different methods to compute the integrals have been explored

- use of IBPs to reduce to master integrals + solution of MIs by differential equations
- use of MB representations to extract pole structure + summation of nested series
- use of sector decomposition

Phase space integrals - methods

Method	Analytical	Numerical
IBP	✔ singly-unresolved integrals ✘ bottleneck is the proliferation of denominators	✔ by evaluating the analytic expressions ✘ no numbers without full analytical results
MB	✔ iterated singly-unresolved integrals ✘ bottleneck is the evaluation of sums	✔ direct numerical evaluation of MB integrals possible ✔ fast and accurate
SD	✔ easy to automate ✘ only in principle, except for lowest order poles	✘ numerical behavior is generally worse than MB method (speed, accuracy)

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

(GS, J. Math. Phys. **52** (2011) 083501.)

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables.

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where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{v} = (v_{11}, v_{12}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{n-1n}, v_{nn}), \qquad v_{kl} \equiv \begin{cases} \dfrac{p_k\cdot p_l}{2} & ; \quad k\neq l \\[2ex] \dfrac{p_k^2}{4} & ; \quad k = l \end{cases}$$

$$\boldsymbol{\alpha} = (\mathbf{0}_N, j_1, \ldots, j_n, 1-j-\epsilon), \qquad \boldsymbol{\beta} = (j_1, \ldots, j_n, 2-j-2\epsilon)$$

and $\mathbf{L_s} = L_{s_1}\times \ldots L_{s_N}$, where L_{s_k} is an infinite contour in the complex s_k-plane running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$.

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where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{A} = \left[\begin{array}{c} -\mathbf{1}_{N\times N} \\ \hline \mathbf{M}_{n\times N} \\ \hline -1\cdots-1 \end{array}\right], \qquad \mathbf{B} = [(0)_{(n+1)\times N}]$$

i.e. **B** is zero, while the $n \times N$ dimensional matrix **M** has the block form:

$$\mathbf{M}_{n\times N} = \left[\ \mathbf{m}_{n\times n}\ \middle|\ \mathbf{m}_{n\times(n-1)}\ \middle|\ \cdots\ \middle|\ \mathbf{m}_{n\times 1}\ \right] \quad \text{with} \quad \mathbf{m}_{n\times p} = \left[\begin{array}{c|c} 0 & (0)_{(n-p)\times(p-1)} \\ \hline 2 & 1\cdots1 \\ \hline 0 & \\ \vdots & \mathbf{1}_{(p-1)\times(p-1)} \\ 0 & \end{array}\right]$$

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

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$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q)\, \frac{1}{(p_1\cdot q)^{j_1}\cdots(p_n\cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)}$$
$$\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^{n}\prod_{l=k}^{n}\frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\,\Gamma(-z_{kl})\,(v_{kl})^{z_{kl}}\right]\left[\prod_{k=1}^{n}\Gamma(j_k+z_k)\right]\Gamma(1-j-\epsilon-z)\,.$$

where

$$z = \sum_{k=1}^{n}\sum_{l=k}^{n} z_{kl}\,, \qquad \text{and} \qquad z_k = \sum_{l=1}^{k} z_{lk} + \sum_{l=k}^{n} z_{kl}\,.$$

Integrated approximate cross sections

Structure of the results

Integrated approximate cross sections

- After summing over unobserved flavors, all integrated approximate cross sections can be written as products (in color space) of various insertion operators with lower point cross sections.

Insertion operators

- color and flavor structure of all insertion operators known
- first two leading poles of kinematical functions entering insertion operators known analytically in all cases (except $\mathbf{I}_2^{(0)}$)
- higher order expansion coefficients computed numerically

Integrated approximate cross sections - an example

MESSAGE

- ➠ Done once and for all (though admittedly lots of tedious work).

Integrated approximate cross sections - an example

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

Each integrable in four dimensions

$$\begin{aligned}
\sigma^{\mathrm{NNLO}}_{m+2} &= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_{m} - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_{m} \right] \right\} \\
\sigma^{\mathrm{NNLO}}_{m+1} &= \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_{m} \right\} \\
\sigma^{\mathrm{NNLO}}_{m} &= \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_{m}
\end{aligned}$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

Integrated approximate cross sections - an example

Iterated singly-unresolved

$$\int_2 \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} = \mathrm{d}\sigma^{\mathrm{B}}_{m} \otimes \mathbf{I}^{(0)}_{12}(\{p\}_m;\epsilon)$$

- structure of insertion operator in color $\otimes$ flavor space

$$\begin{aligned}\mathbf{I}^{(0)}_{12}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\right]^2 \Bigg\{ &\sum_i \left[\mathrm{C}^{(0)}_{12,f_i}\mathbf{T}^2_i + \sum_k \mathrm{C}^{(0)}_{12,f_i f_k}\mathbf{T}^2_k\right]\mathbf{T}^2_i \\ &+ \sum_{j,l}\left[\mathrm{S}^{(0),(j,l)}_{12}C_{\mathrm{A}} + \sum_i \mathrm{CS}^{(0),(j,l)}_{12,f_i}\mathbf{T}^2_i\right]\mathbf{T}_j\mathbf{T}_l \\ &+ \sum_{i,k,j,l}\mathrm{S}^{(0),(i,k)(j,l)}_{12}\{\mathbf{T}_i\mathbf{T}_k,\mathbf{T}_j\mathbf{T}_l\}\Bigg\}\end{aligned}$$

- $\mathrm{C}^{(0)}_{12,f_i}$, $\mathrm{C}^{(0)}_{12,f_i f_k}$, $\mathrm{S}^{(0),(j,l)}_{12}$, $\mathrm{CS}^{(0),(j,l)}_{12,f_i}$ and $\mathrm{S}^{(0),(i,k)(j,l)}_{12}$ are kinematical functions with poles up to $\mathrm{O}(\epsilon^{-4})$ (also depend on PS cut parameters)
- kinematical dependence through

$$x_i = y_{iQ} \equiv \frac{2p_i \cdot Q}{Q^2} \qquad \text{and} \qquad Y_{ik,Q} = \frac{y_{ik}}{y_{iQ}y_{kQ}}$$

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)

$$\mathbf{I}_{12}^{(0)}(p_1,p_2,p_3;\epsilon) = \left[\frac{\alpha_s}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2\left\{\frac{6C_F^2+2C_AC_F+C_A^2}{\epsilon^4}+\left[12C_F^2+\frac{101C_AC_F}{6}\right.\right.$$
$$+\frac{67C_A^2}{12}-\frac{13C_FT_Rn_f}{3}-\frac{3C_AT_Rn_f}{2}-\left(8C_F^2+C_AC_F-\frac{5C_A^2}{2}\right)\ln y_{12}$$
$$-\left(4C_AC_F+\frac{5C_A^2}{2}\right)(\ln y_{13}+\ln y_{23})-(4C_F^2-6C_AC_F-C_A^2)\Sigma(y_0,D_0')$$
$$\left.\left.-(4C_F^2-4C_AC_F)\Sigma(y_0,D_0'-1)\right]\frac{1}{\epsilon^3}+\mathrm{O}(\epsilon^{-2})\right\}$$

- notice x and Y dependence combine to produce just y_{ik} dependence, as expected
- dependence on PS cut parameters through

$$\Sigma(z,N) = \ln z - \sum_{k=1}^{N}\frac{1-(1-z)^k}{k}$$

should vanish once all integrated approximate cross sections are combined

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)

$$\begin{aligned}
\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ & \frac{6C_{\mathrm{F}}^2 + 2C_{\mathrm{A}}C_{\mathrm{F}} + C_{\mathrm{A}}^2}{\epsilon^4} + \Bigg[12C_{\mathrm{F}}^2 + \frac{101C_{\mathrm{A}}C_{\mathrm{F}}}{6} \\
& + \frac{67C_{\mathrm{A}}^2}{12} - \frac{13C_{\mathrm{F}}T_{\mathrm{R}}n_{\mathrm{f}}}{3} - \frac{3C_{\mathrm{A}}T_{\mathrm{R}}n_{\mathrm{f}}}{2} - \left(8C_{\mathrm{F}}^2 + C_{\mathrm{A}}C_{\mathrm{F}} - \frac{5C_{\mathrm{A}}^2}{2}\right)\ln y_{12} \\
& - \left(4C_{\mathrm{A}}C_{\mathrm{F}} + \frac{5C_{\mathrm{A}}^2}{2}\right)(\ln y_{13} + \ln y_{23}) - (4C_{\mathrm{F}}^2 - 6C_{\mathrm{A}}C_{\mathrm{F}} - C_{\mathrm{A}}^2)\Sigma(y_0, D_0') \\
& - (4C_{\mathrm{F}}^2 - 4C_{\mathrm{A}}C_{\mathrm{F}})\Sigma(y_0, D_0' - 1)\Bigg]\frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2})\Bigg\}
\end{aligned}$$

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Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)
- higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \text{O}(\epsilon^1)$$

- kinematical point parametrized by y_{ij}

$$y_{12} = 0.333333, \qquad y_{13} = 0.333333, \qquad y_{23} = 0.333333$$

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	34.12	82.98	34.59	−543.8
$C_\text{A}\,C_\text{F}$	2	9.721	1.209	−142.2	−696.6
C_A^2	1	6.497	12.80	15.87	−47.92
$C_\text{F}\,T_\text{R}\,n_\text{f}$	0	$-\frac{13}{3}$	−32.40	−127.9	−355.2
$C_\text{A}\,T_\text{R}\,n_\text{f}$	0	$-\frac{3}{2}$	−12.01	−46.90	−104.1

Integrated approximate cross sections - an example

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- kinematical point parametrized by y_{ij}

$$y_{12} = 0.238667, \qquad y_{13} = 0.758153, \qquad y_{23} = 0.003180$$

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	36.79	106.0	120.6	-431.0
$C_\text{A}\,C_\text{F}$	2	25.38	143.6	537.3	1505
C_A^2	1	15.24	119.5	660.5	2902
$C_\text{F}\,T_\text{R}\,n_\text{f}$	0	$-\frac{13}{3}$	-31.30	-121.7	-346.0
$C_\text{A}\,T_\text{R}\,n_\text{f}$	0	$-\frac{3}{2}$	-17.72	-109.1	-470.9

Integrated approximate cross sections - an example

Iterated singly-unresolved

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- higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \text{O}(\epsilon^1)$$

- kinematical point parametrized by y_{ij}

$$y_{12} = 0.937044, \qquad y_{13} = 0.024207, \qquad y_{23} = 0.038749$$

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	25.85	34.59	−84.25	−566.8
$C_\text{A}\, C_\text{F}$	2	27.79	136.8	330.6	46.20
C_A^2	1	21.02	195.4	1174	5355
$C_\text{F}\, T_\text{R}\, n_\text{f}$	0	$-\frac{13}{3}$	−57.59	−405.2	−2120
$C_\text{A}\, T_\text{R}\, n_\text{f}$	0	$-\frac{3}{2}$	−24.07	−194.7	−1083

Overview

Counterterm	Types of integrals	Done
$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$	tree level singly-unresolved	✓
$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$	one-loop singly-unresolved	✓
$\int_1 (\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1})^{\mathrm{A}_1}$	tree level iterated singly-unresolved (1)	✓
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$	tree level iterated singly-unresolved (2)	✓
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$	tree level doubly-unresolved	✓/✗

Outlook

Present status

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

Each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \Big] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

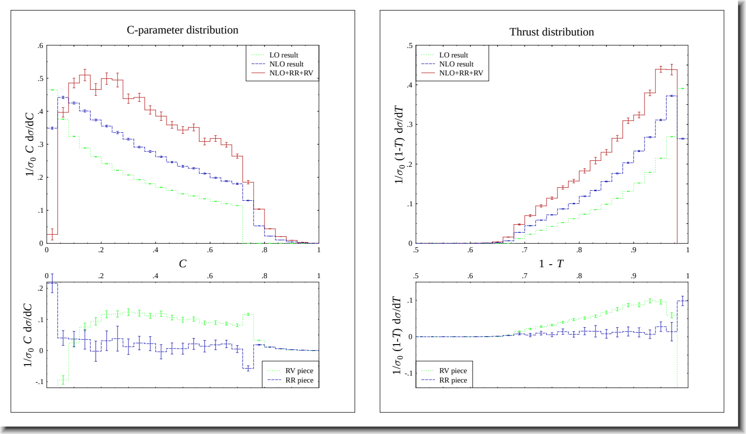
- ✔ unintegrated RR counterterms
- ✔ unintegrated RV counterterms
- ✔ tree-level and one-loop singly-unresolved integrals
- ✔ tree-level iterated singly-unresolved integrals
- ➠ tree-level doubly-unresolved integrals

Present status

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

Numerical Monte Carlo integration (single CPU, ~ 50 hours)



Conclusions

Want to bias you towards

- differential NNLO is a relevant subject
- subtraction is the method of choice for general, efficient calculations

Local subtraction at NNLO

- general, explicit, local subtraction scheme for computing NNLO jet cross sections, for processes with no colored particles in the initial state
- investigated various methods to compute the integrated counterterms: IBP's, MB, SD
- integration of all singly-unresolved and iterated singly-unresolved counterterms finished
- integration of doubly-unresolved counterterms underway

Immediate next steps

- finish doubly-unresolved integrals: only triply-collinear and double soft counterterms left
- consolidate numerics of integrated approximate cross sections

Backup slides

Doubly real counterterms I - IR limits

The doubly real cross section

$$\mathrm{d}\sigma^{\mathrm{RR}}_{m+2} = \mathrm{d}\phi_{m+2}(p_1, \ldots, p_{m+2}) |\mathcal{M}^{(0)}_{m+2}(p_1, \ldots, p_{m+2})|^2$$

Singly-unresolved IR limits

- Collinear: $\mathbf{C}_{ir}$ is a symbolic operator that takes the $p_i || p_r$ limit

$$\mathbf{C}_{ir} |\mathcal{M}^{(0)}_{m+2}(p_i, p_r, \ldots)|^2 = 8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon} \frac{1}{s_{ir}} \langle \mathcal{M}^{(0)}_{m+1}(p_{ir}, \ldots)|\hat{P}_{f_i f_r}(z_i, z_r, k_\perp; \epsilon)|\mathcal{M}^{(0)}_{m+1}(p_{ir}, \ldots)\rangle$$

- Soft: $\mathbf{S}_r$ is a symbolic operator that takes the $p_r \to 0$ limit

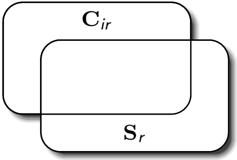
$$\mathbf{S}_r |\mathcal{M}^{(0)}_{m+2}(p_r, \ldots)|^2 = -8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon} \sum_{i,k} \frac{1}{2} \mathcal{S}_{ik}(r) \langle \mathcal{M}^{(0)}_{m+1}(\cancel{p_r}, \ldots)|\mathbf{T}_i \mathbf{T}_k|\mathcal{M}^{(0)}_{m+1}(\cancel{p_r}, \ldots)\rangle$$

NOTICE

- overlap of limits when $p_i || p_r$ and $p_r \to 0$ (matching)
- momenta in factorized matrix elements on r.h.s. conserve momentum only in the strict limit (extension)
- arguments of $\hat{P}_{f_i f_r}$ (z_i, z_r and $k_\perp$) only defined in strict limit (extension)

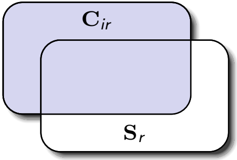
Doubly real counterterms I - matching

Structure of overlaps



Doubly real counterterms I - matching

Structure of overlaps

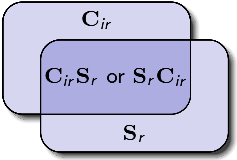


Candidate subtraction term

$$\mathbf{A}_1|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_i \sum_{i\neq r} \frac{1}{2}\mathbf{C}_{ir} \qquad |\mathcal{M}^{(0)}_{m+2}|^2$$

Doubly real counterterms I - matching

Structure of overlaps



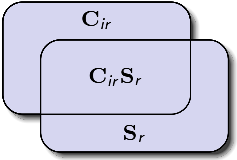
Candidate subtraction term

$$\mathbf{A}_1|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_i \left[\sum_{i\neq r} \frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r \qquad\qquad\qquad \right] |\mathcal{M}^{(0)}_{m+2}|^2$$

- ➠ has correct singularity structure in all limits
- ➠ BUT performs double subtraction when limits overlap (soft-collinear)

Doubly real counterterms I - matching

Structure of overlaps



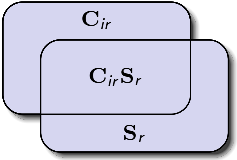
Candidate subtraction term

$$\mathbf{A}_1|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_i \left[\sum_{i\neq r} \frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i\neq r} \mathbf{C}_{ir}\mathbf{S}_r \right] |\mathcal{M}^{(0)}_{m+2}|^2$$

- has correct singularity structure in all limits
- is free of double subtraction
- BUT is still only well-defined in the strict limits

Doubly real counterterms I - matching

Structure of overlaps



$\mathbf{C}_{ir}$

$\mathbf{C}_{ir}\mathbf{S}_r$

$\mathbf{S}_r$

Candidate subtraction term

$$\mathbf{A}_1|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_i \left[\sum_{i\neq r}\frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i\neq r}\mathbf{C}_{ir}\mathbf{S}_r\right]|\mathcal{M}^{(0)}_{m+2}|^2$$

- has correct singularity structure in all limits
- is free of double subtraction
- BUT is still only well-defined in the strict limits

Matching is nontrivial

- collinear and soft limits do not commute: $\mathbf{C}_{ir}\mathbf{S}_r \neq \mathbf{S}_r\mathbf{C}_{ir}$. Hence it is not obvious a priori which limit to remove.

Doubly real counterterms I - extension

Extension over full phase space requires momentum mappings

$$\{p\}_{m+2} \longrightarrow \{\tilde{p}\}_{m+1}$$

- implement exact momentum conservation
- respect the structure of cancellations
- lead to exact phase space factorization

Momentum mappings

- separate collinear and soft momentum mappings
- both distribute recoil momentum democratically
- can be generalized to any number of unresolved momenta

Doubly real counterterms I - extension

Collinear mapping

$$\tilde{p}_{ir}^{\mu} = \frac{1}{1-\alpha_{ir}}(p_i^{\mu} + p_r^{\mu} - \alpha_{ir} Q^{\mu})\,, \qquad \tilde{p}_n^{\mu} = \frac{1}{1-\alpha_{ir}} p_n^{\mu}\,, \quad n \neq i, r$$

$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y_{(ir)Q}^2 - 4y_{ir}}\right]$$

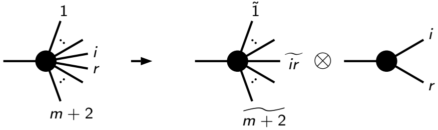
- momentum conservation

$$\tilde{p}_{ir}^{\mu} + \sum_{n\neq i,r} \tilde{p}^{\mu} = p_i^{\mu} + p_r^{\mu} + \sum_{n\neq i,r} p^{\mu}$$

- phase space factorization

$$\mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_{m+1}(\{\tilde{p}\}^{(ir)};Q)[\mathrm{d}p_{1,m+1}^{(ir)}(p_r,\tilde{p}_{ir};Q)]$$

$$[\mathrm{d}p_{1,m+1}^{(ir)}(p_r,\tilde{p}_{ir};Q)] = \mathrm{d}\alpha(1-\alpha)^{2m(1-\epsilon)-1}\frac{s_{\widetilde{ir}Q}}{2\pi}\mathrm{d}\phi_2(p_i,p_r;p_{(ir)})$$



Doubly real counterterms I - extension

Soft mapping

$$\tilde{p}_n^\mu = \Lambda^\mu_\nu[Q,(Q-p_r)/\lambda_r](p_n^\nu/\lambda_r)\,, \quad n \neq r\,, \qquad \lambda_r = \sqrt{1-y_{rQ}}\,,$$

$$\Lambda^\mu_\nu[K,\widetilde{K}] = g^\mu_\nu - \frac{2(K+\widetilde{K})^\mu(K+\widetilde{K})_\nu}{(K+\widetilde{K})^2} + \frac{2K^\mu\widetilde{K}_\nu}{K^2}$$

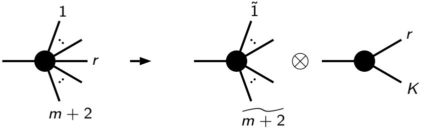
- momentum conservation

$$\sum_{n\neq r}\tilde{p}^\mu = p_r^\mu + \sum_{n\neq r} p^\mu$$

- phase space factorization

$$\mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_{m+1}(\{\tilde{p}\}^{(r)};Q)[\mathrm{d}p^{(r)}_{1,m+1}(p_r;Q)]$$

$$[\mathrm{d}p^{(r)}_{1,m+1}(p_r;Q)] = \mathrm{d}y(1-y)^{m(1-\epsilon)-1}\frac{Q^2}{2\pi}\mathrm{d}\phi_2(p_r,K;Q)$$



Doubly real counterterms I - extension

The collinear and soft momentum mappings define extensions of the limit formulae over the full phase space

$$\mathbf{C}_{ir}|\mathcal{M}_{m+2}^{(0)}|^2 \longrightarrow \mathcal{C}_{ir}^{(0,0)}$$

$$\mathbf{S}_{r}|\mathcal{M}_{m+2}^{(0)}|^2 \longrightarrow \mathcal{S}_{r}^{(0,0)}$$

$$\mathbf{C}_{ir}\mathbf{S}_{r}|\mathcal{M}_{m+2}^{(0)}|^2 \longrightarrow \mathcal{C}_{ir}\mathcal{S}_{r}^{(0,0)}$$

- On the r.h.s. $\mathcal{C}_{ir}^{(0,0)}$, $\mathcal{S}_{r}^{(0,0)}$ and $\mathcal{C}_{ir}\mathcal{S}_{r}^{(0,0)}$ are functions of the original momenta that inherit the notation of the operators, but have nothing to do with taking limits.
- Precise definitions of momentum fractions z_i, z_r and transverse momentum $k_\perp$ that appear in the AP functions are available, but not exhibited.

The true subtraction term

$$\mathbf{A}_1|\mathcal{M}_{m+2}^{(0)}|^2 \longrightarrow \boldsymbol{\mathcal{A}}_1|\mathcal{M}_{m+2}^{(0)}|^2 = \sum_i \left[\sum_{i\neq r}\frac{1}{2}\mathcal{C}_{ir}^{(0,0)} + \mathcal{S}_{r}^{(0,0)} - \sum_{i\neq r}\mathcal{C}_{ir}\mathcal{S}_{r}^{(0,0)}\right]$$

The approximate cross section

$$\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} = \mathrm{d}\phi_{m+1}[\mathrm{d}p_1]\boldsymbol{\mathcal{A}}_1|\mathcal{M}_{m+2}^{(0)}|^2$$

Doubly real counterterms II - IR limits

Doubly-unresolved IR limits

- Triple collinear: $\mathbf{C}_{irs}$ is the operator that takes the $p_i||p_r||p_s$ limit

$$\mathbf{C}_{irs}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s\dots)|^2 = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2\frac{1}{s^2_{irs}}\hat{P}_{f_i f_r f_s}|\mathcal{M}^{(0)}_m(p_{irs},\dots)|^2$$

- Double collinear: $\mathbf{C}_{ir;js}$ is the operator that takes the $p_i||p_r$ and $p_j||p_s$ limit

$$\mathbf{C}_{ir;js}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_j,p_s\dots)|^2 = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2\frac{1}{s_{ir}s_{js}}\hat{P}_{f_i f_r}\hat{P}_{f_j f_s}|\mathcal{M}^{(0)}_m(p_{ir},p_{js},\dots)|^2$$

- Soft-collinear: $\mathbf{CS}_{ir;s}$ is the operator that takes the $p_i||p_r$ and $p_s\to 0$ limit

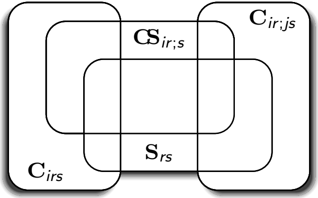
$$\mathbf{CS}_{ir;s}|\mathcal{M}^{(0)}_{m+2}(p_i,p_r,p_s\dots)|^2 = -(8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2\sum_{j,k}\frac{1}{2}\mathcal{S}_{jk}(s)\frac{1}{s_{ir}}\hat{P}_{f_i f_r}|\mathcal{M}^{(0)}_{m,(j,k)}(p_{ir},\cancel{p_s},\dots)|^2$$

- Double soft: $\mathbf{S}_{rs}$ is the operator that takes the $p_r, p_s\to 0$ limit

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+2}(p_r,p_s\dots)|^2 = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2\Bigg[\sum_{i,k,j,l}\frac{1}{8}\mathcal{S}_{ik}(r)\mathcal{S}_{jl}(r)|\mathcal{M}^{(0)}_{m,(i,k),(j,l)}(\cancel{p_r},\cancel{p_s},\dots)|^2 - \frac{1}{4}C_{\mathrm{A}}\sum_{i,k}\mathcal{S}_{ik}(r,s)|\mathcal{M}^{(0)}_{m,(i,k)}(\cancel{p_r},\cancel{p_s},\dots)|^2\Bigg]$$

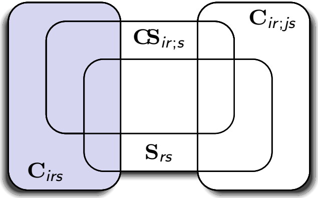
Doubly real counterterms II - matching

Structure of overlaps



Doubly real counterterms II - matching

Structure of overlaps



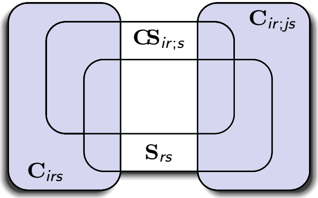
Candidate subtraction term

$$\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_{r}\sum_{s\neq r}\Big\{\sum_{i\neq r,s}\Big[\frac{1}{6}\mathbf{C}_{irs} \qquad\qquad\qquad\qquad \Big]$$

$$\Big\}|\mathcal{M}^{(0)}_{m+2}|^2$$

Doubly real counterterms II - matching

Structure of overlaps



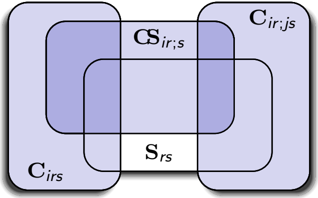
Candidate subtraction term

$$\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_{r}\sum_{s\neq r}\Bigg\{\sum_{i\neq r,s}\Bigg[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s}\frac{1}{8}\mathbf{C}_{ir;js} \qquad\qquad \Bigg]$$

$$\Bigg\}|\mathcal{M}^{(0)}_{m+2}|^2$$

Doubly real counterterms II - matching

Structure of overlaps



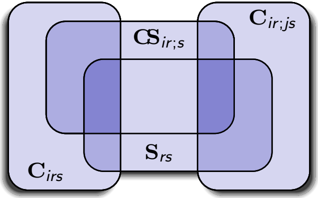
Candidate subtraction term

$$\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_{r}\sum_{s\neq r}\Bigg\{\sum_{i\neq r,s}\Bigg[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s}\frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s}\Bigg]$$

$$\Bigg\}|\mathcal{M}^{(0)}_{m+2}|^2$$

Doubly real counterterms II - matching

Structure of overlaps



Candidate subtraction term

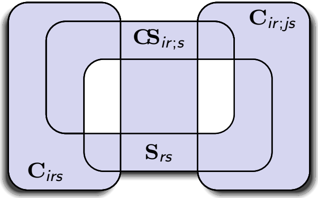
$$\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \Bigg[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s}\Bigg] + \frac{1}{2}\mathbf{S}_{rs}$$

$$\Bigg\}|\mathcal{M}^{(0)}_{m+2}|^2$$

- has correct singularity structure in all limits
- BUT performs double and triple subtraction when limits overlap

Doubly real counterterms II - matching

Structure of overlaps



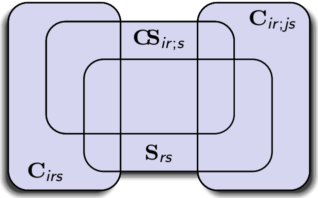
Candidate subtraction term

$$
\begin{aligned}
\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_{r}\sum_{s\neq r}\Bigg\{ & \sum_{i\neq r,s}\left[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s}\frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s}\right] + \frac{1}{2}\mathbf{S}_{rs} \\
& - \sum_{i\neq r,s}\left[\frac{1}{2}\mathbf{C}_{irs}\mathbf{CS}_{ir;s} + \sum_{j\neq i,r,s}\frac{1}{2}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} + \frac{1}{2}\mathbf{C}_{irs}\mathbf{S}_{rs} + \mathbf{CS}_{ir;s}\mathbf{S}_{rs}\right. \\
& \left. + \sum_{j\neq i,r,s}\frac{1}{2}\mathbf{C}_{ir;js}\mathbf{S}_{rs}\right] \Bigg\}|\mathcal{M}^{(0)}_{m+2}|^2
\end{aligned}
$$

- has correct singularity structure in all limits
- BUT performs double and triple subtraction when limits overlap

Doubly real counterterms II - matching

Structure of overlaps



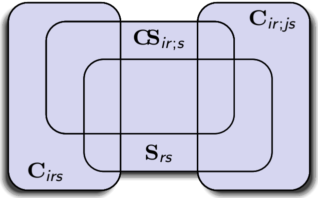
Candidate subtraction term

$$\begin{aligned}
\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \sum_{s\neq r} \Bigg\{ & \sum_{i\neq r,s} \Bigg[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s} \Bigg] + \frac{1}{2}\mathbf{S}_{rs} \\
& - \sum_{i\neq r,s} \Bigg[\frac{1}{2}\mathbf{C}_{irs}\mathbf{CS}_{ir;s} + \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} + \frac{1}{2}\mathbf{C}_{irs}\mathbf{S}_{rs} + \mathbf{CS}_{ir;s}\mathbf{S}_{rs} \\
& + \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{S}_{rs} \Bigg] + \sum_{i\neq r,s} \Bigg[\mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} + \sum_{j\neq i,r,s} \mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} \Bigg] \Bigg\} |\mathcal{M}^{(0)}_{m+2}|^2
\end{aligned}$$

- has correct singularity structure in all limits
- is free of double subtraction
- BUT is still only well-defined in the strict limits

Doubly real counterterms II - matching

Structure of overlaps



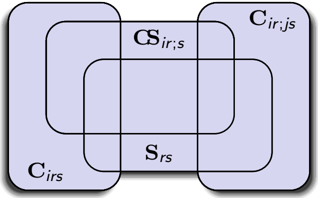
Candidate subtraction term

$$\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \Bigg[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s} \Bigg] + \frac{1}{2}\mathbf{S}_{rs}$$
$$- \sum_{i\neq r,s} \Bigg[\frac{1}{2}\mathbf{C}_{irs}\mathbf{CS}_{ir;s} + \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} + \frac{1}{2}\mathbf{C}_{irs}\mathbf{S}_{rs} + \mathbf{CS}_{ir;s}\mathbf{S}_{rs}$$
$$+ \sum_{j\neq i,r,s} \frac{1}{2}\boxed{\mathbf{C}_{ir;js}\mathbf{S}_{rs}} \Bigg] + \sum_{i\neq r,s} \Bigg[\mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} + \sum_{j\neq i,r,s} \boxed{\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\mathbf{S}_{rs}} \Bigg] \Bigg\} |\mathcal{M}^{(0)}_{m+2}|^2$$

- has correct singularity structure in all limits
- is free of double subtraction
- BUT is still only well-defined in the strict limits

Doubly real counterterms II - matching

Structure of overlaps



Candidate subtraction term

$$\begin{aligned}\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2 = \sum_{r}\sum_{s\neq r}\Bigg\{&\sum_{i\neq r,s}\left[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s}\frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s}\right] + \frac{1}{2}\mathbf{S}_{rs}\\ &- \sum_{i\neq r,s}\left[\frac{1}{2}\mathbf{C}_{irs}\mathbf{CS}_{ir;s} + \sum_{j\neq i,r,s}\frac{1}{2}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} + \frac{1}{2}\mathbf{C}_{irs}\mathbf{S}_{rs} + \mathbf{CS}_{ir;s}\mathbf{S}_{rs}\right.\\ &\left.- \sum_{j\neq i,r,s}\frac{1}{2}\mathbf{C}_{ir;js}\mathbf{S}_{rs} - \mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs}\right]\Bigg\}|\mathcal{M}_{m+2}^{(0)}|^2\end{aligned}$$

- has correct singularity structure in all limits
- is free of double subtraction
- BUT is still only well-defined in the strict limits

Doubly real counterterms II - extension

Extension over full phase space requires momentum mappings

$$\{p\}_{m+2} \longrightarrow \{\tilde{p}\}_m$$

- ➠ implement exact momentum conservation
- ➠ respect the structure of cancellations
- ➠ lead to exact phase space factorization

Momentum mappings

- ➠ separate triple collinear, double collinear, soft-collinear and double soft momentum mappings
- ➠ all distribute recoil momentum democratically
- ➠ generalizations of singly-unresolved mappings, details not exhibited here

Doubly real counterterms II - extension

The various momentum mappings define extensions of the limit formulae over the full phase space

$$
\begin{aligned}
\mathbf{C}_{irs}|\mathcal{M}_{m+2}^{(0)}|^2 &\longrightarrow \mathcal{C}_{irs}^{(0,0)} \\
\mathbf{C}_{ir;js}|\mathcal{M}_{m+2}^{(0)}|^2 &\longrightarrow \mathcal{C}_{ir;js}^{(0,0)} \\
\mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}|^2 &\longrightarrow \mathcal{CS}_{ir;s}^{(0,0)} \\
\mathbf{S}_{rs}|\mathcal{M}_{m+2}^{(0)}|^2 &\longrightarrow \mathcal{S}_{rs}^{(0,0)} \\
&\vdots
\end{aligned}
$$

- On the r.h.s. $\mathcal{C}_{ir}^{(0,0)}$, $\mathcal{S}_{r}^{(0,0)}$ and $\mathcal{C}_{ir}\mathcal{S}_{r}^{(0,0)}$ are functions of the original momenta that inherit the notation of the operators, but have nothing to do with taking limits.
- Precise definitions of momentum fractions z_i, z_r and transverse momentum $k_\perp$ that appear in the AP functions are available, but not exhibited.

Doubly real counterterms II - extension

The true subtraction term

$$\boldsymbol{\mathcal{A}}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \Bigg[\frac{1}{6}\mathcal{C}^{(0,0)}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathcal{C}^{(0,0)}_{ir;js} + \frac{1}{2}\mathcal{CS}^{(0,0)}_{ir;s} \Bigg] + \frac{1}{2}\mathcal{S}^{(0,0)}_{rs}$$
$$- \sum_{i\neq r,s} \Bigg[\frac{1}{2}\mathcal{C}_{irs}\mathcal{CS}^{(0,0)}_{ir;s} + \sum_{j\neq i,r,s} \frac{1}{2}\mathcal{C}_{ir;js}\mathcal{CS}^{(0,0)}_{ir;s} + \frac{1}{2}\mathcal{C}_{irs}\mathcal{S}^{(0,0)}_{rs} + \mathcal{CS}_{ir;s}\mathcal{S}^{(0,0)}_{rs}$$
$$- \sum_{j\neq i,r,s} \frac{1}{2}\mathcal{C}_{ir;js}\mathcal{S}^{(0,0)}_{rs} - \mathcal{C}_{irs}\mathcal{CS}_{ir;s}\mathcal{S}^{(0,0)}_{rs} \Bigg] \Bigg\}$$

The approximate cross section

$$\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} = \mathrm{d}\phi_m[\mathrm{d}p_2]\boldsymbol{\mathcal{A}}_2|\mathcal{M}^{(0)}_{m+2}|^2$$

Doubly real counterterms III

Overlap of singly- and doubly-unresolved approximate cross sections

- $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$ is the singly-unresolved approximation to $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$

$$\boldsymbol{\mathcal{A}}_{12}|\mathcal{M}_{m+2}^{(0)}|^2 = \sum_i \left[\sum_{i\neq r} \frac{1}{2}\mathcal{C}_{ir}\boldsymbol{\mathcal{A}}_2 + \mathcal{S}_r\boldsymbol{\mathcal{A}}_2 - \sum_{i\neq r}\mathcal{C}_{ir}\mathcal{S}_r\boldsymbol{\mathcal{A}}_2\right]|\mathcal{M}_{m+2}^{(0)}|^2$$

- The functions $\mathcal{C}_{ir}\boldsymbol{\mathcal{A}}_2$, $\mathcal{S}_r\boldsymbol{\mathcal{A}}_2$ and $\mathcal{C}_{ir}\mathcal{S}_r\boldsymbol{\mathcal{A}}_2$ are sums of terms, all precisely defined, not shown.
- Extensions use iterated singly-unresolved momentum mappings.

The approximate cross section

$$\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} = \mathrm{d}\phi_m[\mathrm{d}p_1][\mathrm{d}p_1]\boldsymbol{\mathcal{A}}_{12}|\mathcal{M}_{m+2}^{(0)}|^2$$

NONTRIVIAL

- $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$ correctly cancels the overlap between $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$ and $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$.

Real-virtual counterterms

Real-virtual approximate cross sections constructed in the same way

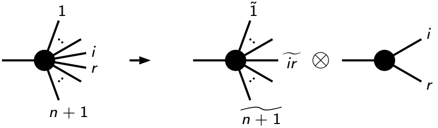
- The real-virtual approximate cross section $\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$ is constructed exactly like $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$, only the specific IR limit formulae change.

Singly-unresolved integrals

Collinear phase space factorization

$$\mathrm{d}\phi_{n+1}(\{p\};Q) = \mathrm{d}\phi_n(\{\tilde{p}\}^{(ir)};Q)[\mathrm{d}p_{1,n}^{(ir)}(p_r,\tilde{p}_{ir};Q)]$$

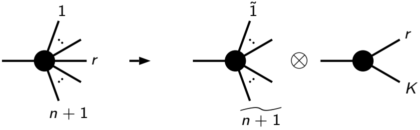
$$[\mathrm{d}p_{1,n}^{(ir)}(p_r,\tilde{p}_{ir};Q)] = \mathrm{d}\alpha(1-\alpha)^{2(n-1)(1-\epsilon)-1}\frac{s_{\widetilde{ir}Q}}{2\pi}\mathrm{d}\phi_2(p_i,p_r;p_{(ir)})$$



Soft phase space factorization

$$\mathrm{d}\phi_{n+1}(\{p\};Q) = \mathrm{d}\phi_n(\{\tilde{p}\}^{(r)};Q)[\mathrm{d}p_{1,n}^{(r)}(p_r;Q)]$$

$$[\mathrm{d}p_{1,n}^{(r)}(p_r;Q)] = \mathrm{d}y(1-y)^{(n-1)(1-\epsilon)-1}\frac{Q^2}{2\pi}\mathrm{d}\phi_2(p_r,K;Q)$$



Collinear integrals

Collinear counterterms defined using Altarelli-Parisi splitting functions

$$\int_0^{\alpha_0} \mathrm{d}\alpha (1-\alpha)^{2d_0-1} \frac{s_{\widetilde{ir}Q}}{2\pi} \int \mathrm{d}\phi_2(p_i, p_r; p_{(ir)}) \frac{1}{s_{ir}^{1+\kappa\epsilon}} P_{f_i f_r}^{(\kappa)}(z_i, z_r; \epsilon)$$

Collinear integrals

Collinear counterterms defined using Altarelli-Parisi splitting functions

$$\int_0^{\alpha_0} \mathrm{d}\alpha (1-\alpha)^{2d_0-1} \frac{s_{\widetilde{ir}Q}}{2\pi} \int \boxed{\mathrm{d}\phi_2(p_i, p_r; p_{(ir)})} \frac{1}{s_{ir}^{1+\kappa\epsilon}} P^{(\kappa)}_{f_i f_r}(z_i, z_r; \epsilon)$$

- explicit parametrization of PS gives ($x = 2\tilde{p}_{ir} \cdot Q/Q^2$)

$$\mathrm{d}\phi_2(p_i, p_r; p_{(ir)}) = \frac{s_{ir}^{-\epsilon}}{8\pi} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \mathrm{d}s_{ir}\, \mathrm{d}v\, \delta\Big\{ s_{ir} - Q^2[\alpha(\alpha + (1-\alpha)x)] \Big\}$$
$$\cdot [v(1-v)]^{-\epsilon} \Theta(v) \Theta(1-v)$$

Collinear integrals

Collinear counterterms defined using Altarelli-Parisi splitting functions

$$\int_0^{\alpha_0} \mathrm{d}\alpha (1-\alpha)^{2d_0-1} \frac{s_{\widetilde{ir}\,Q}}{2\pi} \int \mathrm{d}\phi_2(p_i, p_r; p_{(ir)}) \boxed{\frac{1}{s_{ir}^{1+\kappa\epsilon}} P_{f_i f_r}^{(\kappa)}(z_i, z_r; \epsilon)}$$

➠ Altarelli-Parisi functions can be expressed as linear combinations of

$$\frac{z_r^{k+\delta\epsilon}}{s_{ir}^{1+\kappa\epsilon}} g_l^{\pm}(z_r)\,, \qquad z_r = \frac{p_r \cdot Q}{(p_i + p_r)\cdot Q} = \frac{\alpha + (1-\alpha)xv}{2\alpha + (1-\alpha)x}$$

with $k = -1, 0, 1, 2$ and

δ	Function	$g_l^{\pm}(z)$
0	g_A	1
∓ 1	$g_B^{(\pm)}$	$(1-z)^{\pm\epsilon}$
0	$g_C^{(\pm)}$	$(1-z)^{\pm\epsilon}{}_2F_1(\pm\epsilon, \pm\epsilon, 1\pm\epsilon, z)$
± 1	$g_D^{(\pm)}$	${}_2F_1(\pm\epsilon, \pm\epsilon, 1\pm\epsilon, z)$

Soft integrals

Soft counterterms defined using eikonal functions (and its collinear limit)

$$\int_0^{y_0} \mathrm{d}y(1-y)^{d_0'-1} \frac{Q^2}{2\pi} \int \mathrm{d}\phi_2(p_r, K; Q) \left\{ \left(\frac{s_{ik}}{s_{ir} s_{kr}} \right)^{1+\kappa\epsilon}, \left(\frac{1}{s_{ir}} \frac{z_i}{z_r} \right)^{1+\kappa\epsilon} \right\}$$

Soft integrals

Soft counterterms defined using eikonal functions (and its collinear limit)

$$\int_0^{y_0} \mathrm{d}y (1-y)^{d_0'-1} \frac{Q^2}{2\pi} \int \mathrm{d}\phi_2(p_r, K; Q) \left\{ \left(\frac{s_{ik}}{s_{ir} s_{kr}} \right)^{1+\kappa\epsilon}, \left(\frac{1}{s_{ir}} \frac{z_i}{z_r} \right)^{1+\kappa\epsilon} \right\}$$

- choose a frame

$$Q^\mu = \sqrt{s}(1, \ldots), \qquad \tilde{p}_i^\mu = \tilde{E}_i(1, \ldots, 1), \qquad \tilde{p}_k^\mu = \tilde{E}_i(1, \ldots, \sin\chi, \cos\chi)$$

and

$$p_r^\mu = E_r(1, ..\text{‘angles’}.., \sin\vartheta \sin\varphi, \cos\vartheta)$$

to find the explicit parametrization of PS

$$\mathrm{d}\phi_2(p_r, K; Q) = \frac{Q^{-2\epsilon}}{16\pi^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \mathrm{d}\varepsilon_r \, \varepsilon_r^{1-2\epsilon} \delta(\varepsilon_r - y)$$
$$\cdot \, \mathrm{d}(\cos\vartheta)(\sin\vartheta)^{-2\epsilon} \mathrm{d}(\cos\varphi)(\sin\varphi)^{-1-2\epsilon}$$

Soft integrals

Soft counterterms defined using eikonal functions (and its collinear limit)

$$\int_0^{y_0} \mathrm{d}y (1-y)^{d_0'-1} \frac{Q^2}{2\pi} \int \mathrm{d}\phi_2(p_r, K; Q) \left\{ \boxed{\left(\frac{s_{ik}}{s_{ir} s_{kr}}\right)^{1+\kappa\epsilon}, \left(\frac{1}{s_{ir}} \frac{z_i}{z_r}\right)^{1+\kappa\epsilon}} \right\}$$

- precise definition of the soft momentum mapping implies

$$s_{ik} = (1-\varepsilon_r) s_{\tilde{i}\tilde{k}}\,, \qquad s_{ir} = s_{\tilde{i}r}\,, \qquad s_{kr} = s_{\tilde{k}r}\,, \qquad s_{iQ} = (1-\varepsilon_r) s_{\tilde{i}Q} + s_{\tilde{i}r}$$

hence ($\cos\chi = 1 - 2Y$)

$$\frac{s_{ik}}{s_{ir} s_{kr}} = \frac{4Y}{Q^2} \frac{1-\varepsilon_r}{\varepsilon_r^2} \frac{1}{(1-\cos\vartheta)(1-\cos\chi\cos\vartheta - \sin\chi\sin\vartheta\cos\varphi)}$$

$$\frac{1}{s_{ir}} \frac{z_i}{z_r} = \frac{1}{Q^2} \frac{1}{\varepsilon_r} \left[1 + \frac{2(1-\varepsilon_r)}{\varepsilon_r(1-\cos\vartheta)}\right]$$

Basic one-particle integrals

Computed (semi)analytically

(Aglietti, Del Duca, Duhr, Trócsányi, GS;
Bolzoni, Moch, Trócsányi, GS)

- collinear type

$$\mathcal{I} \propto x \int_0^{\alpha_0} \mathrm{d}\alpha\, \alpha^{-1-(1+\kappa)\epsilon} (1-\alpha)^{2d_0-1} [\alpha + (1-\alpha)x]^{-1-(1+\kappa)\epsilon}$$

$$\cdot \int_0^1 \mathrm{d}v [v\,(1-v)]^{-\epsilon} \left(\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x}\right)^{k+\delta\epsilon} g_l^{(\pm)}\left(\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x}\right)$$

- soft type

$$\mathcal{J} \propto -Y^{1+\kappa\epsilon} \int_0^{y_0} \mathrm{d}y\, y^{-1-2(1+\kappa)\epsilon}(1-y)^{d_0'+\kappa\epsilon} \int_{-1}^1 \mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon}$$

$$\cdot \int_{-1}^1 \mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon} \frac{1}{[(1-\cos\vartheta)(1-\cos\chi\cos\vartheta - \sin\chi\sin\vartheta\cos\varphi)]^{1+\kappa\epsilon}}$$

$$\mathcal{K} \propto \int_0^{y_0} \mathrm{d}y\, y^{-(2+\kappa)\epsilon}(1-y)^{d_0'-1} \int_{-1}^1 \mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon} \left[1+\frac{2(1-y)}{y(1-\cos\vartheta)}\right]^{1+\kappa\epsilon}$$

Iterated singly-unresolved integrals

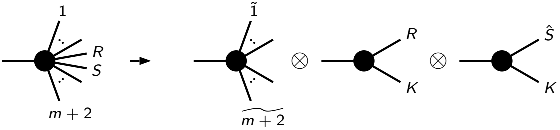
The momentum mapping is of iterated form

$$\{p\}_{m+2} \to \{\hat{p}\}_{m+1} \to \{\tilde{p}\}_m$$

Phase space factorization (schematically)

$$\mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_m(\{\tilde{p}\}^{(\hat{S},R)};Q)[\mathrm{d}p^{(R)}_{1,m+1}][\mathrm{d}p^{(\hat{S})}_{1,m}]$$

factorized phase space measures from singly-unresolved case



Examples of iterated one-particle integrals

Collinear-double collinear counterterm from $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2}$ (Bolzoni, Trócsányi, GS)

$$\mathcal{C}_{kt}\mathcal{C}^{(0)}_{ir;kt} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{kt}}\frac{1}{\widehat{s}_{ir}} \langle \mathcal{M}^{(0)}_m(\{\tilde{p}\})| P^{(0)}_{f_k f_t}(z_{t,k};\epsilon) P^{(0)}_{f_i f_r}(\widehat{z}_{r,i};\epsilon)|\mathcal{M}^{(0)}_m(\{\tilde{p}\})\rangle$$
$$\cdot\,(1-\alpha_{kt})^{2d_0-2m(1-\epsilon)}(1-\widehat{\alpha}_{kt})^{2d_0-2m(1-\epsilon)}\Theta(\alpha_0-\alpha_{kt})\Theta(\alpha_0-\widehat{\alpha}_{ir})$$

Using the discussed explicit parametrization of the factorized PS measures, we find its integral can be expressed as a linear combination of the following “MIs”

$$\mathcal{I}^{(4)}_{\mathcal{C}}(x_k,x_i;\epsilon,\alpha_0,d_0,k,l) = x_k x_i \int_0^{\alpha_0}\mathrm{d}\alpha \int_0^{\alpha_0}\mathrm{d}\beta\,\alpha^{-1-\epsilon}(1-\alpha)^{2d_0-1}$$
$$\cdot\,\beta^{-1-\epsilon}(1-\beta)^{2d_0-2+2\epsilon}[\alpha+(1-\alpha)(1-\beta)x_k]^{-1-\epsilon}[\beta+(1-\beta)x_i]^{-1-\epsilon}$$
$$\cdot\int_0^1\mathrm{d}v\int_0^1\mathrm{d}u\,v^{-\epsilon}(1-v)^{-\epsilon}u^{-\epsilon}(1-u)^{-\epsilon}$$
$$\cdot\left(\frac{\alpha+(1-\alpha)(1-\beta)x_k v}{2\alpha+(1-\alpha)(1-\beta)x_k}\right)^k\left(\frac{\beta+(1-\beta)x_i u}{2\beta+(1-\beta)x_i}\right)^l$$

where $k,l=-1,0,1,2$.

Examples of iterated one-particle integrals

Abelian soft-double soft counterterm from $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2}$ (Bolzoni, Trócsányi, GS)

$$\left(\mathcal{S}_t\mathcal{S}^{(0)}_{rt}\right)^{\mathrm{ab}} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \sum_{i,j,k,l}\frac{1}{8}\mathcal{S}_{\hat{i}\hat{k}}(\hat{r})\mathcal{S}_{jl}(t)|\mathcal{M}^{(0)}_{m,(i,k)(j,l)}(\{\tilde{p}\})|^2$$

$$\cdot\,(1-y_{tQ})^{d_0'-m(1-\epsilon)}(1-y_{\hat{r}Q})^{d_0'-m(1-\epsilon)}\Theta(y_0-y_{tQ})\Theta(y_0-y_{\hat{r}Q})$$

Consider e.g. $j=i$ and $l=k$. Using the discussed explicit parametrization of the factorized PS measures, we find that $[\mathrm{S}_t\mathrm{S}^{(0)}_{rt}]_{ikik}$ is proportional to

$$\mathcal{I}^{(11)}_{\mathcal{S}}(Y_{ik,Q};\epsilon,y_0,d_0') = -\frac{4\Gamma^4(1-\epsilon)}{\pi\Gamma^2(1-\epsilon)}\,\frac{B_{y_0}(-2\epsilon,d_0'+1)}{\epsilon}\,Y_{ik,Q}\int_0^{y_0}\mathrm{d}y\,y^{-1-2\epsilon}(1-y)^{d_0'-1+\epsilon}$$

$$\cdot\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1}\mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon}\left[f(\vartheta,\varphi;0)\right]^{-1}\left[f(\vartheta,\varphi;Y_{ik,Q})\right]^{-1}$$

$$\cdot\left[Y(y,\vartheta,\varphi;Y_{ik,Q})\right]^{-\epsilon}{}_2F_1\big(-\epsilon,-\epsilon,1-\epsilon,1-Y(y,\vartheta,\varphi;Y_{ik,Q})\big)$$

where

$$f(\vartheta,\varphi;Y_{ik,Q}) = 1-2\sqrt{Y_{ik,Q}(1-Y_{ik,Q})}\sin\vartheta\cos\varphi-(1-2Y_{ik,Q})\chi\cos\vartheta$$

$$Y(y,\vartheta,\varphi;\chi) = \frac{4(1-y)Y_{ik,Q}}{[2(1-y)+y\,f(\vartheta,\varphi;0)][2(1-y)+y\,f(\vartheta,\varphi;Y_{ik,Q})]}\,.$$

Doubly-unresolved integrals

Genuine doubly-unresolved momentum mapping

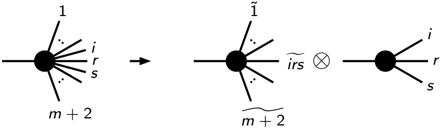
$$\{p\}_{m+2} \to \{\tilde{p}\}_m$$

Phase space factorization (schematically)

$$\mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_m(\{\tilde{p}\}^{(RS)};Q)[\mathrm{d}p_{2,m}^{(RS)}]$$

factorized phase space measure for (e.g.) the triple collinear mapping is

$$[\mathrm{d}p_{2;m}^{(irs)}(p_r,p_s,\tilde{p}_{irs};Q)] = \mathrm{d}\alpha(1-\alpha)^{2(m-1)(1-\epsilon)-1}\frac{s_{\widetilde{irs}\,Q}}{2\pi}\mathrm{d}\phi_3(p_i,p_r,p_s;p_{(irs)})$$



An example of a two-particle integral

Triple collinear counterterm from $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}}$

$$\mathcal{C}_{irs}^{(0,0)} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{irs}^2} \langle \mathcal{M}_m^{(0)}(\{\tilde{p}\})| P_{f_i f_r f_s}^{(0)}(\{z_{j,kl}, s_{jk}\};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\})\rangle$$
$$\cdot (1-\alpha_{irs})^{2d_0-2m(1-\epsilon)}\Theta(\alpha_0-\alpha_{irs})$$

Using an explicit parametrization of the factorized PS measure, we find its integral can be expressed as a linear combination of the following "MIs"

$$\mathcal{I}(x;\epsilon,\alpha_0,d_0,n_1,n_2,n_3,n_4,n_5,n_6,n_7) = \frac{\Gamma^2(1-\epsilon)}{\pi\Gamma(1-2\epsilon)} x \int_0^{\alpha_0} \mathrm{d}\alpha\, \alpha^{-1-2\epsilon}(1-\alpha)^{2d_0-3+2\epsilon}$$
$$\cdot [\alpha+(1-\alpha)x]^{-1-\epsilon} \int_0^1 \mathrm{d}v_r\,\mathrm{d}v_s\,\mathrm{d}t_{ir}\,\mathrm{d}t_{is}\,\mathrm{d}t_{rs}\,\delta(1-t_{ir}-t_{is}-t_{rs})(1-t_{ir})(1-t_{is})$$
$$\cdot [(t_{rs}^+-t_{rs})(t_{rs}-t_{rs}^-)]^{-\frac{1}{2}-\epsilon}\Theta(t_{rs}^+-t_{rs})\Theta(t_{rs}-t_{rs}^-) t_{ir}^{n_1} t_{is}^{n_2} t_{rs}^{n_3}(1-t_{ir})^{n_4}(1-t_{is})^{n_5}$$
$$\cdot \left(\frac{\alpha+(1-\alpha)xv_r}{2\alpha+(1-\alpha)x}\right)^{n_6} \left(\frac{\alpha+(1-\alpha)xv_s}{2\alpha+(1-\alpha)x}\right)^{n_7}$$

where

$$t_{rs}^{\pm} = (1-t_{ir})(1-t_{is})\left[\sqrt{v_r(1-v_s)} \pm \sqrt{v_s(1-v_r)}\right]^2 .$$

More integrated approximate cross sections 1

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

Each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

More integrated approximate cross sections 1

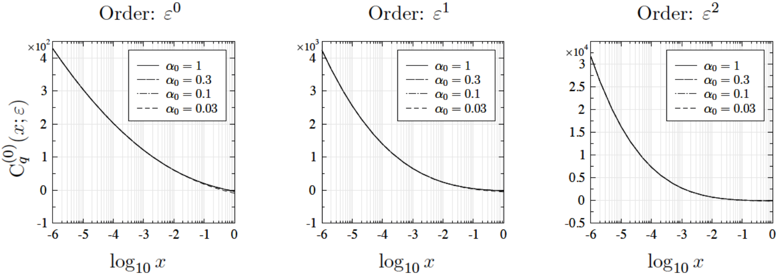
Tree-level singly-unresolved

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \mathbf{I}^{(0)}_1(\{p\}_{m+1};\epsilon)$$

➠ The insertion operator has the following structure

$$\mathbf{I}^{(0)}_1(\{p\}_{m+1};\epsilon) = \frac{\alpha_\mathrm{s}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon \sum_i \left[\mathrm{C}^{(0)}_{1,f_i}\mathbf{T}^2_i + \sum_k \mathrm{S}^{(0),(i,k)}_1 \mathbf{T}_i\mathbf{T}_k\right]$$

- Pole structure of $\mathbf{I}^{(0)}_1$ coincides with known result
- In this case higher order expansion coefficients known analytically



More integrated approximate cross sections 1

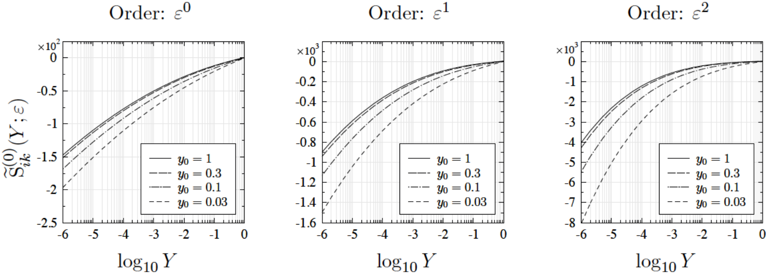
Tree-level singly-unresolved

$$\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} = \mathrm{d}\sigma^{\mathrm{R}}_{m+1} \otimes \boldsymbol{\mathrm{I}}^{(0)}_1(\{p\}_{m+1};\epsilon)$$

➠ The insertion operator has the following structure

$$\boldsymbol{\mathrm{I}}^{(0)}_1(\{p\}_{m+1};\epsilon) = \frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon \sum_i \left[\mathrm{C}^{(0)}_{1,f_i} \boldsymbol{\mathrm{T}}^2_i + \sum_k \mathrm{S}^{(0),(i,k)}_1 \boldsymbol{\mathrm{T}}_i \boldsymbol{\mathrm{T}}_k\right]$$

- Pole structure of $\boldsymbol{\mathrm{I}}^{(0)}_1$ coincides with known result
- In this case higher order expansion coefficients known analytically



More integrated approximate cross sections 2

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

Each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

More integrated approximate cross sections 2

One-loop singly-unresolved

$$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} = \mathrm{d}\sigma_m^{\mathrm{V}} \otimes \mathbf{I}_1^{(0)}(\{p\}_m;\epsilon) + \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \mathbf{I}_1^{(1)}(\{p\}_m;\epsilon)$$

- Notice $\mathbf{I}_1^{(0)}(\{p\}_m;\epsilon)$ is the same insertion operator as at tree level
- The one-loop insertion operator has the following structure

$$\mathbf{I}_1^{(1)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \frac{\Gamma^2(1+\epsilon)\Gamma^4(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)} \cdot \sum_i \left[\mathrm{C}_{1,f_i}^{(1)} \mathbf{T}_i^2 + \sum_k \mathrm{S}_1^{(1),(i,k)} \mathbf{T}_i \mathbf{T}_k + \sum_{k,l} \mathrm{S}_1^{(1),(i,k,l)} f_{abc} T_i^a T_k^b T_l^c\right]$$

  - Pole structure of $\mathbf{I}_1^{(1)}$ known analytically up to $\mathrm{O}(\epsilon^{-1})$
  - Higher order expansion coefficients computed numerically

More integrated approximate cross sections 2

Approximation to integrated singly-unresolved

$$\int_1 \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \left[\frac{1}{2} \left\{ \mathbf{I}_1^{(0)}(\{p\}_m;\epsilon), \mathbf{I}_1^{(0)}(\{p\}_m;\epsilon) \right\} + \mathbf{I}_1^{R\times(0)}(\{p\}_m;\epsilon) \right]$$

- Notice $\mathbf{I}_1^{(0)}(\{p\}_m;\epsilon)$ is the same insertion operator as at tree level
- $\mathbf{I}_1^{R\times(0)}(\{p\}_m;\epsilon)$ part has the same structure as $\mathbf{I}_1^{(0)}(\{p\}_m;\epsilon)$

$$\mathbf{I}_1^{R\times(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2} \right)^\epsilon \right]^2 \sum_i \left[\mathrm{C}_{1,f_i}^{R\times(0)} \mathbf{T}_i^2 + \sum_k \mathrm{S}_1^{R\times(0),(i,k)} \mathbf{T}_i \mathbf{T}_k \right]$$

  - Pole structure of $\mathbf{I}_1^{R\times(0)}$ known analytically up to $\mathrm{O}(\epsilon^{-1})$
  - Higher order expansion coefficients computed numerically

More integrated approximate cross sections 3

Doubly-unresolved

$$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \mathbf{I}_2^{(0)}(\{p\}_m;\epsilon)$$

- The insertion operator has the same structure as $\mathbf{I}_{12}^{(0)}(\{p\}_m;\epsilon)$

$$\mathbf{I}_2^{(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_\mathrm{s}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ \sum_i \left[\mathrm{C}_{2,f_i}^{(0)} \mathbf{T}_i^2 + \sum_k \mathrm{C}_{2,f_i f_k}^{(0)} \mathbf{T}_k^2\right] \mathbf{T}_i^2$$

$$+ \sum_{j,l} \left[\mathrm{S}_2^{(0),(j,l)} C_\mathrm{A} + \sum_i \mathrm{CS}_{2,f_i}^{(0),(j,l)} \mathbf{T}_i^2\right] \mathbf{T}_j \mathbf{T}_l$$

$$+ \sum_{i,k,j,l} \mathrm{S}_2^{(0),(i,k)(j,l)} \{\mathbf{T}_i \mathbf{T}_k, \mathbf{T}_j \mathbf{T}_l\} \Bigg\}$$

  - Work in progress

More on methods: IBP

1. Algebraic reduction of the integrand by means of partial fractioning
$$\frac{1}{x(1-x)(1-xyz)} = \frac{1}{x} + \frac{1}{1-yz}\frac{1}{1-x} + \frac{y^2z^2}{1-yz}\frac{1}{1-xyz}$$
Note the appearance of a new denominator: $1-yz$. With increasing numbers of variables, the number of new denominators grows very rapidly.
2. Reduction to master integrals by means of IBP identities. We can use the standard Laporta algorithm to solve the IBP relations, but we find the occurrence of surface terms in the IBPs, consisting of integrals of lower dimensionality than the original ones.
3. Analytical evaluation of the master integrals. We obtain the ϵ expansion of the MIs by solving systems of differential equations, expanded in ϵ. The final results contain one- and two-dimensional harmonic polylogarithms. For some MIs, a nontrivial basis extension of 2dHPLs is necessary.

More on methods: MB

1. Convert sums into products in the integrand

$$\frac{1}{(a+b)^\nu} = \frac{1}{\Gamma(\nu)} \int_{q-\mathrm{i}\infty}^{q+\mathrm{i}\infty} \frac{\mathrm{d}z}{2\pi\mathrm{i}} a^{-\nu-z} b^z \Gamma(\nu+z)\Gamma(-\nu)$$

2. Integrate over the real variables to obtain MB integrals

$$(1-x)^p = \int_0^1 \mathrm{d}y\, y^p \delta(1-x-y)$$

$$\int_0^1 \mathrm{d}x\,\mathrm{d}y\, x^{p_1} y^{p_2} \delta(1-x-y) = \frac{\Gamma(p_1)\Gamma(p_2)}{\Gamma(p_1+p_2)}$$

3. Resolve the pole structure by shifting integration contours.
4. Compute the MB integrals, converting them into sums over residua.
5. Perform the sums.

More on methods: SD

Example

1. Transform the integral so that the range of integration is the unit hypercube, and all singularities are at the borders.

$$I = \int_0^1 \mathrm{d}x\,\mathrm{d}y\, x^{-1-\epsilon} y^{-\epsilon} [x + (1-x)y]^{-1}$$

2. Decompose into "sectors" using $1 = [\Theta(x-y) + \Theta(y-x)]$
3. Remap each integration region to the unit hypercube: for $x \geq y$ set $y \to xt$, for $y \geq x$ set $x \to yt$.

$$\begin{aligned} I &= \int_0^1 \mathrm{d}x\,\mathrm{d}t\, x^{-1-2\epsilon} t^{-\epsilon} [1 + (1-x)t]^{-1} \\ &+ \int_0^1 \mathrm{d}t\,\mathrm{d}y\, t^{-1-\epsilon} y^{-1-2\epsilon} [1 + (1-y)t]^{-1} \end{aligned}$$

4. Resolve the pole structure using simple residuum subtraction. This gives a finite integral representation for the expansion coefficients.
5. Integrate these representations.

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables.

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

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where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{v} = (v_{11}, v_{12}, \dots, v_{1n}, v_{22}, v_{23}, \dots, v_{n-1n}, v_{nn}), \qquad v_{kl} \equiv \begin{cases} \dfrac{p_k\cdot p_l}{2} & ; \quad k \neq l \\ \dfrac{p_k^2}{4} & ; \quad k = l \end{cases}$$

$$\boldsymbol{\alpha} = (\mathbf{0}_N, j_1, \dots, j_n, 1-j-\epsilon), \qquad \boldsymbol{\beta} = (j_1, \dots, j_n, 2-j-2\epsilon)$$

and $\mathbf{L_s} = L_{s_1} \times \ldots L_{s_N}$, where L_{s_k} is an infinite contour in the complex s_k-plane running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$.

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q)\,\frac{1}{(p_1\cdot q)^{j_1}\cdots(p_n\cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v};(\boldsymbol{\alpha},\mathbf{A});(\boldsymbol{\beta},\mathbf{B});\mathbf{L_s}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{A} = \left[\begin{array}{c} -\mathbf{1}_{N\times N} \\ \hline \mathbf{M}_{n\times N} \\ \hline -1\cdots-1 \end{array}\right], \qquad \mathbf{B} = [(0)_{(n+1)\times N}]$$

i.e. $\mathbf{B}$ is zero, while the $n\times N$ dimensional matrix $\mathbf{M}$ has the following block form:

$$\mathbf{M}_{n\times N} = \left[\begin{array}{c|c|c|c} \mathbf{m}_{n\times n} & \mathbf{m}_{n\times(n-1)} & \cdots & \mathbf{m}_{n\times 1}\end{array}\right] \quad \text{with} \quad \mathbf{m}_{n\times p} = \left[\begin{array}{c|c} 0 & (0)_{(n-p)\times(p-1)} \\ \hline 2 & 1\cdots 1 \\ \hline 0 & \\ \vdots & \mathbf{1}_{(p-1)\times(p-1)} \\ 0 & \end{array}\right]$$

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_s}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\}; \epsilon) = 2^{2-j-2\epsilon} \pi^{1-\epsilon} \frac{1}{\prod_{k=1}^{n} \Gamma(j_k) \Gamma(2-j-2\epsilon)}$$
$$\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left[\prod_{k=1}^{n} \prod_{l=k}^{n} \frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}} \Gamma(-z_{kl}) \, (v_{kl})^{z_{kl}} \right] \left[\prod_{k=1}^{n} \Gamma(j_k + z_k) \right] \Gamma(1-j-\epsilon-z) \,.$$

where

$$z = \sum_{k=1}^{n} \sum_{l=k}^{n} z_{kl} \,, \qquad \text{and} \qquad z_k = \sum_{l=1}^{k} z_{lk} + \sum_{l=k}^{n} z_{kl} \,.$$

Angular Integral with n Denominators – Definition

Motivated by the previous example, we define d dimensional angular integrals with n denominators as follows

Definition (angular integral with n denominators)

Let $p_1^\mu, \dots, p_n^\mu$ be some fixed vectors in d dimensional Minkowski space. Then

$$\Omega_{j_1,\dots,j_n} \equiv \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

where $\mathrm{d}\Omega_{d-1}(q)$ is the angular measure in d dimensions for the massless vector q^μ.

Note that $\Omega_{j_1,\dots,j_n}$ is clearly rotationally invariant and the overall normalization of the p_i^μ and q^μ plays no essential role.

Angular Integral with n Denominators – Definition (the small print)

More precisely (but less to the point), we may choose a Lorentz frame and normalization such that

$$p_1^\mu = (1, \mathbf{0}_{d-2}, \beta_1)\,, \qquad p_2^\mu = (1, \mathbf{0}_{d-3}, \beta_2 \sin\chi_2^{(1)}, \beta_2 \cos\chi_2^{(1)})$$

$$\vdots$$

$$p_n^\mu = (1, \mathbf{0}_{d-1-n}, \beta_n \prod_{k=1}^{n-1} \sin\chi_n^{(k)}, \beta_n \cos\chi_n^{(n-1)} \prod_{k=1}^{n-2} \sin\chi_n^{(k)}, \ldots, \beta_n \cos\chi_n^{(2)} \sin\chi_n^{(1)}, \beta_n \cos\chi_n^{(1)})$$

$$q^\mu = (1, ..\text{`angles'}.., \cos\vartheta_n \prod_{k=1}^{n-1} \sin\vartheta_k, \cos\vartheta_{n-1} \prod_{k=1}^{n-2} \sin\vartheta_k, \ldots, \cos\vartheta_2 \sin\vartheta_1, \cos\vartheta_1)$$

Definition (angular integral with n denominators)

$$\Omega_{j_1,\ldots,j_n} \equiv \int \mathrm{d}\Omega_{d-1-n}(q) \int_{-1}^{1} \left[\prod_{k=1}^{n} \mathrm{d}(\cos\vartheta_k)\,(\sin\vartheta_k)^{-k+1-2\epsilon}\right]$$

$$\times \prod_{k=1}^{n} \left\{1 - \beta_k \sum_{l=1}^{k} \left[\left(\delta_{lk} + (1-\delta_{lk}) \cos\chi_k^{(l)}\right) \cos\vartheta_l \prod_{m=1}^{l-1} \left(\sin\chi_k^{(m)} \sin\vartheta_m\right)\right]\right\}^{-j_k}$$

where we used

$$\mathrm{d}\Omega_{d-1}(q) = \prod_{k=1}^{n} \mathrm{d}(\cos\vartheta_k)\,(\sin\vartheta_k)^{-k+1-2\epsilon} \mathrm{d}\Omega_{d-1-n}(q)$$

Angular Integral with n Denominators – Definition

Motivated by the previous example, we define d dimensional angular integrals with n denominators as follows

Definition (angular integral with n denominators)

Let $p_1^\mu, \ldots, p_n^\mu$ be some fixed vectors in d dimensional Minkowski space. Then

$$\Omega_{j_1,\ldots,j_n} \equiv \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

where $\mathrm{d}\Omega_{d-1}(q)$ is the angular measure in d dimensions for the massless vector q^μ.

Note that the definition is more general than it seems at first sight, e.g.

$$p_1^\mu = (1, \mathbf{0}_{d-2}, \beta_1), \qquad q^\mu = (1, ..\text{'angles'}.., \cos\vartheta)$$

then

$$(p_1 + q)^2 = p_1^2 + 2p_1 \cdot q = (1 - \beta_1^2) + 2(1 - \beta_1 \cos\vartheta) = (3 - \beta_1^2)\left[1 - \frac{2\beta_1}{3 - \beta_1^2}\cos\vartheta\right]$$

$$= (3 - \beta_1^2)(p_1' \cdot q)$$

where

$$p_1'^{\,\mu} = \left(1, \mathbf{0}_{d-2}, \frac{2\beta_1}{3 - \beta_1^2}\right), \qquad q^\mu = (1, ..\text{'angles'}.., \cos\vartheta)$$

Angular Integral with n Denominators – General Result 1

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

where $H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$ is the H-function of the following $N = \frac{n(n+1)}{2}$ variables

$$\mathbf{v} = (v_{11}, v_{12}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{n-1n}, v_{nn}), \qquad v_{kl} \equiv \begin{cases} \dfrac{p_k \cdot p_l}{2} & ; \quad k \neq l \\ \dfrac{p_k^2}{4} & ; \quad k = l \end{cases}$$

We are assuming the general case, i.e. $v_{kl} > 0$, $\forall\, k, l = 1, \ldots, n$.

(The parameters $(\boldsymbol{\alpha}, \mathbf{A})$, $(\boldsymbol{\beta}, \mathbf{B})$ as well as $\mathbf{L_S}$ will be discussed momentarily.)

The result looks tidy enough, but you might be wondering what exactly an H-function of several variables is.

The *H*-function of several variables

Definition (*H*-function of *N* variables)

$$H[\mathbf{x};(\boldsymbol{\alpha},\mathbf{A});(\boldsymbol{\beta},\mathbf{B});\mathbf{L_s}] \equiv (2\pi\mathrm{i})^{-N}\int_{\mathbf{L_s}} \Theta(\mathbf{s})\,\mathbf{x^s}\,\mathbf{ds}$$

where

$$\Theta(\mathbf{s}) = \frac{\prod_{j=1}^{m}\Gamma\left(\alpha_j + \sum_{k=1}^{N} a_{j,k}s_k\right)}{\prod_{j=1}^{n}\Gamma\left(\beta_j + \sum_{k=1}^{N} b_{j,k}s_k\right)}$$

Here $\mathbf{s} = (s_1,\ldots,s_N)$, $\mathbf{x} = (x_1,\ldots,x_N)$, $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_m)$ and $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ denote vectors of complex numbers; while

$$\mathbf{A} = (a_{j,k})_{m\times N} \qquad \text{and} \qquad \mathbf{B} = (b_{j,k})_{n\times N}$$

are matrices of real numbers. Also

$$\mathbf{x^s} = \prod_{k=1}^{N}(x_k)^{s_k}\,; \qquad \mathbf{ds} = \prod_{k=1}^{N}\mathrm{d}s_k\,; \qquad \mathbf{L_s} = L_{s_1}\times\ldots\times L_{s_N}\,,$$

where L_{s_k} is an infinite contour in the complex s_k-plane running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ such that $\Theta(\mathbf{s})$ has no singularities for $\mathbf{s}\in\mathbf{L_s}$.

[N. T. Hai, H. M. Srivastava, Computers Math. Applic. **29**, 17 (1995)]

The *H*-function of several variables

Some comments

- The H-function of several variables generalizes nearly all known special functions of N variables, e.g. Lauricella functions $F_A^{(N)}$, $F_B^{(N)}$, $F_C^{(N)}$ and $F_D^{(N)}$; the G-function of N variables; the special H-function of N variables, etc.
- For the specific cases of $N = 1$ and 2, it essentially reduces to the known Fox's H-function of one variable and the H-function of two variables defined by various authors scattered in the literature.
- The H-function of several variables satisfies various contiguous relations, i.e. algebraic relations between functions $H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$ with the vectors of parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ shifted by vectors of integers. These relations may be used to reduce H-functions to a set of basis functions with parameters differing form the original values by integer shifts.

  [O. P. Tandon, Indian J. Pure Appl. Math. **11**, 321 (1980)]

  [C. M. Joshi, J. P. Arya, Indian J. Pure Appl. Math. **12**, 826 (1981)]

  [O. P. Tandon, Indian J. Math. **24**, 55 (1982)]
- The definition given above is different form the H-function considered by Hai and Srivastava only in the replacement of $\mathbf{x}^{-\mathbf{S}}$ by $\mathbf{x}^{\mathbf{S}}$. We have made this replacement for convenience in our applications.

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v};(\boldsymbol{\alpha},\mathbf{A});(\boldsymbol{\beta},\mathbf{B});\mathbf{L_S}]$$

where $H[\mathbf{v};(\boldsymbol{\alpha},\mathbf{A});(\boldsymbol{\beta},\mathbf{B});\mathbf{L_S}]$ is the H-function of the following $N = \frac{n(n+1)}{2}$ variables

$$\mathbf{v} = (v_{11}, v_{12}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{n-1n}, v_{nn}), \qquad v_{kl} \equiv \begin{cases} \dfrac{p_k \cdot p_l}{2} & ; \quad k \neq l \\[2ex] \dfrac{p_k^2}{4} & ; \quad k = l \end{cases}$$

We are assuming the general case, i.e. $v_{kl} > 0$, $\forall\, k, l = 1, \ldots, n$.

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the following vectors of parameters

$$\boldsymbol{\alpha} = (\mathbf{0}_N, j_1, \ldots, j_n, 1 - j - \epsilon)\,, \qquad \boldsymbol{\beta} = (j_1, \ldots, j_n, 2 - j - 2\epsilon)$$

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v};(\boldsymbol{\alpha},\mathbf{A});(\boldsymbol{\beta},\mathbf{B});\mathbf{L_s}]$$

A and **B** are $\frac{(n+1)(n+2)}{2} \times N$ and $(n+1) \times N$ matrices of parameters, respectively

$$\mathbf{A} = \left[\begin{array}{c} -\mathbf{1}_{N\times N} \\ \hline \mathbf{M}_{n\times N} \\ \hline -1 \cdots -1 \end{array}\right], \qquad \mathbf{B} = [(0)_{(n+1)\times N}]$$

i.e. **B** is zero, while the $n \times N$ dimensional matrix **M** has the following block form

$$\mathbf{M}_{n\times N} = \left[\begin{array}{c|c|c|c} \mathbf{m}_{n\times n} & \mathbf{m}_{n\times(n-1)} & \cdots & \mathbf{m}_{n\times 1} \end{array}\right] \quad \text{with} \quad \mathbf{m}_{n\times p} = \left[\begin{array}{c|c} 0 & (0)_{(n-p)\times(p-1)} \\ \hline 2 & 1\cdots 1 \\ \hline 0 & \\ \vdots & \mathbf{1}_{(p-1)\times(p-1)} \\ 0 & \end{array}\right]$$

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_s}]$$

A and **B** are $\frac{(n+1)(n+2)}{2} \times N$ and $(n+1) \times N$ matrices of parameters, respectively e.g.

$$\mathbf{A}_1 = \left[\begin{array}{c} -1 \\ \hline 2 \\ \hline -1 \end{array}\right], \quad \mathbf{A}_2 = \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ \hline 2 & 1 & \multicolumn{1}{|c}{0} \\ 0 & 1 & \multicolumn{1}{|c}{2} \\ \hline -1 & -1 & -1 \end{array}\right], \quad \mathbf{A}_3 = \left[\begin{array}{cccccc} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 2 & 1 & 1 & \multicolumn{1}{|c}{0} & 0 & \multicolumn{1}{|c}{0} \\ 0 & 1 & 0 & \multicolumn{1}{|c}{2} & 1 & \multicolumn{1}{|c}{0} \\ 0 & 0 & 1 & \multicolumn{1}{|c}{0} & 1 & \multicolumn{1}{|c}{2} \\ \hline -1 & -1 & -1 & -1 & -1 & -1 \end{array}\right]$$

Angular Integral with n Denominators – General Result 2

The d dimensional angular integrals with n denominators can be evaluated in terms of the so-called H-function of several variables.

Result

Let $j = j_1 + \ldots + j_n$. Then we have

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

Hence we find the Mellin–Barnes representation

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} \frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)}$$

$$\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left[\prod_{k=1}^{n}\prod_{l=k}^{n}\frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})(v_{kl})^{z_{kl}}\right]\left[\prod_{k=1}^{n}\Gamma(j_k+z_k)\right]\Gamma(1-j-\epsilon-z)\,.$$

where the contours of integration for z_{kl} are chosen such that poles with $\Gamma(\ldots + z_{kl})$ dependence are to the left of the contour and poles with $\Gamma(\ldots - z_{kl})$ dependence are to the right of it.

Analytical results – one denominator

- One denominator, massless ($p_1^2 = 0$)

$$\Omega_j(0;\epsilon) = \int \mathrm{d}\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} \frac{\Gamma(1-j-\epsilon)}{\Gamma(2-j-2\epsilon)}$$

- One denominator, massive ($p_1^2 = 4v_{11} \neq 0$)

$$\Omega_j(v_{11};\epsilon) = \int \mathrm{d}\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j} = 2^{2-2\epsilon} \pi^{1-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \times {}_2F_1\left(\frac{j}{2}, \frac{j+1}{2}, \frac{3}{2} - \epsilon, 1 - 4v_{11}\right)$$

Analytical results – two denominators

- Two denominators, massless ($p_1^2 = p_2^2 = 0$, $p_1 \cdot p_2 = 2v_{12} \neq 0$)

$$\Omega_{j,k}(v_{12}, 0, 0; \epsilon) = \int \mathrm{d}\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j (p_2 \cdot q)^k} = 2^{2-j-k-2\epsilon} \pi^{1-\epsilon} \frac{\Gamma(1-j-\epsilon)\Gamma(1-k-\epsilon)}{\Gamma(1-\epsilon)\Gamma(2-j-k-2\epsilon)} \times {}_2F_1(j, k, 1-\epsilon, 1-v_{12})$$

- Two denominators, one mass ($p_1^2 = 4v_{11} \neq 0$, $p_2^2 = 0$, $p_1 \cdot p_2 = 2v_{12} \neq 0$)

$$\Omega_{j,k}(v_{12}, v_{11}, 0; \epsilon) = \int \mathrm{d}\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j (p_2 \cdot q)^k} = 2^{2-j-k-2\epsilon} \pi^{1-\epsilon} \frac{\Gamma(1-k-\epsilon)}{\Gamma(2-k-2\epsilon)} v_{12}^{-j} \times F_1\left(j, 1-k-\epsilon, 1-k-\epsilon, 2-k-2\epsilon, \frac{2v_{12}-1-\sqrt{1-4v_{11}}}{2v_{12}}, \frac{2v_{12}-1+\sqrt{1-4v_{11}}}{2v_{12}}\right)$$